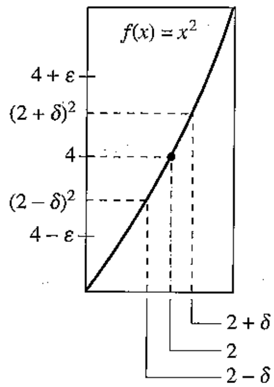


The limit of $f(x)$ as x approaches 2 is 4.
Figure 1.15

Example 8 Using the ε-δ Definition of a Limit

Use the ε-δ definition of a limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ when } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval $(1, 3)$, you know that $|x + 2| < 5$. So, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Throughout this chapter you will use the ε-δ definition of a limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of a limit.

EXERCISES FOR SECTION 1.2

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4. $\displaystyle\lim_{x \to -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$						

5. $\displaystyle\lim_{x \to 3} \frac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x \to 4} \frac{[x/(x + 1)] - (4/5)}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

7. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

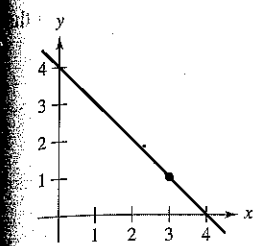
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

8. $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x}$

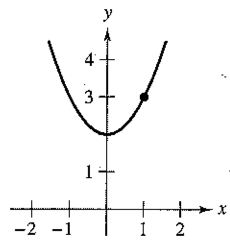
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 9–18, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

9. $\lim\limits_{x\to 3} (4 - x)$

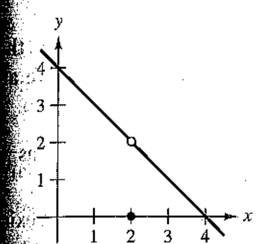


10. $\lim\limits_{x\to 1} (x^2 + 2)$



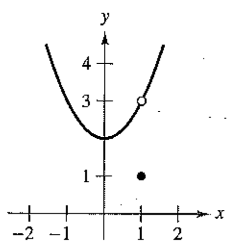
11. $\lim\limits_{x\to 2} f(x)$

$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$

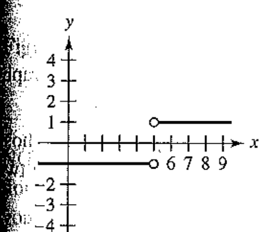


12. $\lim\limits_{x\to 1} f(x)$

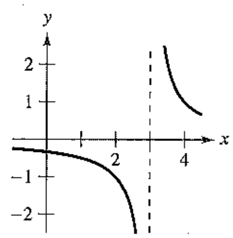
$f(x) = \begin{cases} x^2 + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$



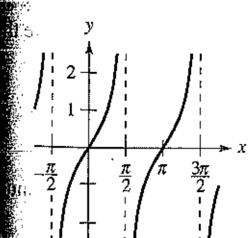
13. $\lim\limits_{x\to 5} \dfrac{|x - 5|}{x - 5}$



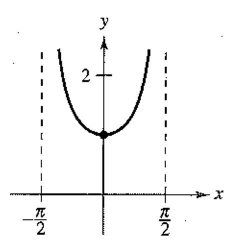
14. $\lim\limits_{x\to 3} \dfrac{1}{x - 3}$



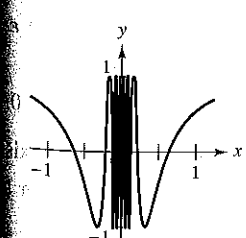
15. $\lim\limits_{x\to \pi/2} \tan x$



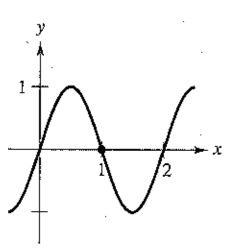
16. $\lim\limits_{x\to 0} \sec x$



17. $\lim\limits_{x\to 0} \cos \dfrac{1}{x}$



18. $\lim\limits_{x\to 1} \sin \pi x$



19. *Modeling Data* The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute. A formula for the cost is given by

$$C(t) = 0.75 - 0.50 [\![-(t - 1)]\!]$$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \leq x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \leq 5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t\to 3.5} C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

20. Repeat Exercise 19 if $C(t) = 0.35 - 0.12[\![-(t - 1)]\!]$.

21. The graph of $f(x) = 2 - 1/x$ is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$ then $|f(x) - 1| < 0.1$.

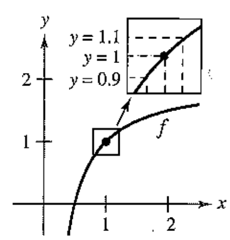


Figure for 21

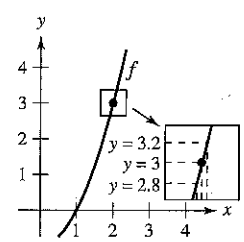


Figure for 22

22. The graph of $f(x) = x^2 - 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$ then $|f(x) - 3| < 0.2$.

In Exercises 23–26, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

23. $\lim\limits_{x\to 2} (3x + 2)$

24. $\lim\limits_{x\to 4} \left(4 - \dfrac{x}{2}\right)$

25. $\lim\limits_{x\to 2} (x^2 - 3)$

26. $\lim\limits_{x\to 5} (x^2 + 4)$

In Exercises 27–38, find the limit L. Then use the ε-δ definition to prove that the limit is L.

27. $\lim_{x \to 2} (x + 3)$
28. $\lim_{x \to -3} (2x + 5)$
29. $\lim_{x \to -4} \left(\frac{1}{2}x - 1\right)$
30. $\lim_{x \to 1} \left(\frac{2}{3}x + 9\right)$
31. $\lim_{x \to 6} 3$
32. $\lim_{x \to 2} (-1)$
33. $\lim_{x \to 0} \sqrt[3]{x}$
34. $\lim_{x \to 4} \sqrt{x}$
35. $\lim_{x \to -2} |x - 2|$
36. $\lim_{x \to 3} |x - 3|$
37. $\lim_{x \to 1} (x^2 + 1)$
38. $\lim_{x \to -3} (x^2 + 3x)$

Writing **In Exercises 39–42, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

39. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

$\lim_{x \to 4} f(x)$

40. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x \to 3} f(x)$

41. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x \to 9} f(x)$

42. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x \to 3} f(x)$

Getting at the Concept

43. Write a brief description of the meaning of the notation $\lim_{x \to 8} f(x) = 25$.
44. (a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.
    (b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.
45. Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.
46. Determine the limit of the function describing the atmospheric pressure on a plane as it descends from 32,000 feet to land at Honolulu, located at sea level. (The atmospheric pressure at sea level is 14.7 lb/in.^{2}.)

47. Consider the function $f(x) = (1 + x)^{1/x}$. Estimate the limit

$$\lim_{x \to 0} (1 + x)^{1/x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

48. *Graphical Analysis* The statement

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

True or False? **In Exercises 49–52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

49. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.
50. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.
51. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.
52. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

53. ***Programming*** Use the programming capabilities of a graphing utility to write a program for approximating $\lim_{x \to c} f(x)$. Assume the program will be applied only to functions whose limits exist as x approaches c. Let $y_1 = f(x)$ and generate two lists whose entries form the ordered pairs

$$(c \pm [0.1]^n,\ f(c \pm [0.1]^n))$$

for $n = 0, 1, 2, 3,$ and 4.

54. Use the program you created in Exercise 53 to approximate the limit

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}.$$

55. Prove that if the limit of $f(x)$ as $x \to c$ exists, then the limit must be unique. [*Hint:* Let

$$\lim_{x \to c} f(x) = L_1 \quad \text{and} \quad \lim_{x \to c} f(x) = L_2$$

and prove that $L_1 = L_2$.]

56. Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of a limit to prove that $\lim_{x \to c} f(x) = mc + b$.
57. Prove that $\lim_{x \to c} f(x) = L$ is equivalent to $\lim_{x \to c} [f(x) - L] = 0$.
58. Given that $\lim_{x \to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

Lab Series LAB 2

EXERCISES FOR SECTION 1.3

In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = x^2 - 5x$

(a) $\lim_{x\to 5} h(x)$

(b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim_{x\to 4} g(x)$

(b) $\lim_{x\to 0} g(x)$

3. $f(x) = x\cos x$

(a) $\lim_{x\to 0} f(x)$

(b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$

(a) $\lim_{t\to 4} f(t)$

(b) $\lim_{t\to -1} f(t)$

In Exercises 5–22, find the limit.

5. $\lim_{x\to 2} x^4$

6. $\lim_{x\to -2} x^3$

7. $\lim_{x\to 0} (2x - 1)$

8. $\lim_{x\to -3} (3x + 2)$

9. $\lim_{x\to -3} (x^2 + 3x)$

10. $\lim_{x\to 1} (-x^2 + 1)$

11. $\lim_{x\to -3} (2x^2 + 4x + 1)$

12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$

13. $\lim_{x\to 2} \dfrac{1}{x}$

14. $\lim_{x\to -3} \dfrac{2}{x + 2}$

15. $\lim_{x\to 1} \dfrac{x - 3}{x^2 + 4}$

16. $\lim_{x\to 3} \dfrac{2x - 3}{x + 5}$

17. $\lim_{x\to 7} \dfrac{5x}{\sqrt{x + 2}}$

18. $\lim_{x\to 3} \dfrac{\sqrt{x + 1}}{x - 4}$

19. $\lim_{x\to 3} \sqrt{x + 1}$

20. $\lim_{x\to 4} \sqrt[3]{x + 4}$

21. $\lim_{x\to -4} (x + 3)^2$

22. $\lim_{x\to 0} (2x - 1)^3$

In Exercises 23–26, find the limits.

23. $f(x) = 5 - x,\ g(x) = x^3$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$

(a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$

(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x\to \pi/2} \sin x$

28. $\lim_{x\to \pi} \tan x$

29. $\lim_{x\to 2} \cos\dfrac{\pi x}{3}$

30. $\lim_{x\to 1} \sin\dfrac{\pi x}{2}$

31. $\lim_{x\to 0} \sec 2x$

32. $\lim_{x\to \pi} \cos 3x$

33. $\lim_{x\to 5\pi/6} \sin x$

34. $\lim_{x\to 5\pi/3} \cos x$

35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$

36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x\to c} f(x) = 2$

$\lim_{x\to c} g(x) = 3$

(a) $\lim_{x\to c} [5g(x)]$

(b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = \frac{3}{2}$

$\lim_{x\to c} g(x) = \frac{1}{2}$

(a) $\lim_{x\to c} [4f(x)]$

(b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$

(a) $\lim_{x\to c} [f(x)]^3$

(b) $\lim_{x\to c} \sqrt{f(x)}$

(c) $\lim_{x\to c} [3f(x)]$

(d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$

(a) $\lim_{x\to c} \sqrt[3]{f(x)}$

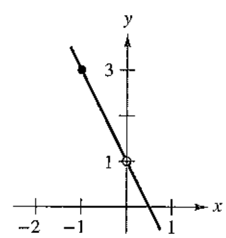
(b) $\lim_{x\to c} \dfrac{f(x)}{18}$

(c) $\lim_{x\to c} [f(x)]^2$

(d) $\lim_{x\to c} [f(x)]^{2/3}$

In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

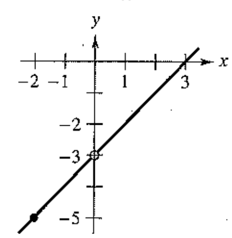
41. $g(x) = \dfrac{-2x^2 + x}{x}$



(a) $\lim_{x\to 0} g(x)$

(b) $\lim_{x\to -1} g(x)$

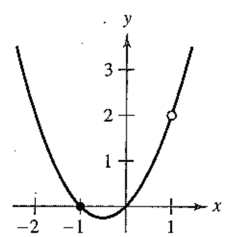
42. $h(x) = \dfrac{x^2 - 3x}{x}$



(a) $\lim_{x\to -2} h(x)$

(b) $\lim_{x\to 0} h(x)$

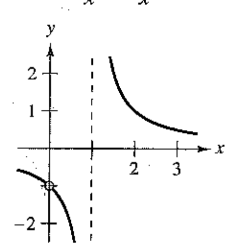
43. $g(x) = \dfrac{x^3 - x}{x - 1}$



(a) $\lim_{x\to 1} g(x)$

(b) $\lim_{x\to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x\to-1} \frac{x^2-1}{x+1}$

46. $\lim_{x\to-1} \frac{2x^2-x-3}{x+1}$

47. $\lim_{x\to2} \frac{x^3-8}{x-2}$

48. $\lim_{x\to-1} \frac{x^3+1}{x+1}$

In Exercises 49–62, find the limit (if it exists).

49. $\lim_{x\to5} \frac{x-5}{x^2-25}$

50. $\lim_{x\to2} \frac{2-x}{x^2-4}$

51. $\lim_{x\to-3} \frac{x^2+x-6}{x^2-9}$

52. $\lim_{x\to4} \frac{x^2-5x+4}{x^2-2x-8}$

53. $\lim_{x\to0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$

54. $\lim_{x\to0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

55. $\lim_{x\to4} \frac{\sqrt{x+5}-3}{x-4}$

56. $\lim_{x\to3} \frac{\sqrt{x+1}-2}{x-3}$

57. $\lim_{x\to0} \frac{[1/(3+x)]-(1/3)}{x}$

58. $\lim_{x\to0} \frac{[1/(x+4)]-(1/4)}{x}$

59. $\lim_{\Delta x\to0} \frac{2(x+\Delta x)-2x}{\Delta x}$

60. $\lim_{\Delta x\to0} \frac{(x+\Delta x)^2-x^2}{\Delta x}$

61. $\lim_{\Delta x\to0} \frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

62. $\lim_{\Delta x\to0} \frac{(x+\Delta x)^3-x^3}{\Delta x}$

Graphical, Numerical, and Analytic Analysis **In Exercises 63–66, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

63. $\lim_{x\to0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

64. $\lim_{x\to16} \frac{4-\sqrt{x}}{x-16}$

65. $\lim_{x\to0} \frac{[1/(2+x)]-(1/2)}{x}$

66. $\lim_{x\to2} \frac{x^5-32}{x-2}$

In Exercises 67–78, determine the limit of the trigonometric function (if it exists).

67. $\lim_{x\to0} \frac{\sin x}{5x}$

68. $\lim_{x\to0} \frac{3(1-\cos x)}{x}$

69. $\lim_{x\to0} \frac{\sin x(1-\cos x)}{2x^2}$

70. $\lim_{\theta\to0} \frac{\cos\theta\tan\theta}{\theta}$

71. $\lim_{x\to0} \frac{\sin^2 x}{x}$

72. $\lim_{x\to0} \frac{\tan^2 x}{x}$

73. $\lim_{h\to0} \frac{(1-\cos h)^2}{h}$

74. $\lim_{\phi\to\pi} \phi\sec\phi$

75. $\lim_{x\to\pi/2} \frac{\cos x}{\cot x}$

76. $\lim_{x\to\pi/4} \frac{1-\tan x}{\sin x-\cos x}$

77. $\lim_{t\to0} \frac{\sin 3t}{2t}$

78. $\lim_{x\to0} \frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x\to0}\left(\frac{2\sin 2x}{2x}\right)\left(\frac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 79–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

79. $\lim_{t\to0} \frac{\sin 3t}{t}$

80. $\lim_{h\to0} (1+\cos 2h)$

81. $\lim_{x\to0} \frac{\sin x^2}{x}$

82. $\lim_{x\to0} \frac{\sin x}{\sqrt[3]{x}}$

In Exercises 83–86, find $\lim_{\Delta x\to0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

83. $f(x) = 2x+3$

84. $f(x) = \sqrt{x}$

85. $f(x) = \frac{4}{x}$

86. $f(x) = x^2-4x$

In Exercises 87 and 88, use the Squeeze Theorem to find $\lim_{x\to c} f(x)$.

87. $c = 0$
$4 - x^2 \le f(x) \le 4 + x^2$

88. $c = a$
$b - |x-a| \le f(x) \le b + |x-a|$

In Exercises 89–94, use a graphing utility to graph the given function and the equations $y = |x|$ and $y = -|x|$ in the same viewing window. Using the graphs to visually observe the Squeeze Theorem, find $\lim_{x\to0} f(x)$.

89. $f(x) = x\cos x$

90. $f(x) = |x\sin x|$

91. $f(x) = |x|\sin x$

92. $f(x) = |x|\cos x$

93. $f(x) = x\sin\frac{1}{x}$

94. $h(x) = x\cos\frac{1}{x}$

Getting at the Concept

95. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.
96. Give an example of two functions that agree at all but one point.
97. What is meant by an indeterminate form?
98. In your own words, explain the Squeeze Theorem.

99. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to0} h(x) = 1.$$

100. ***Writing*** Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin^2 x, \quad \text{and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x \to 0} h(x) = 0.$$

Free-Falling Object **In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 1000$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 1000 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

101. If a construction worker drops a wrench from a height of 1000 feet, how fast will the wrench be falling after 5 seconds?

102. If a construction worker drops a wrench from a height of 1000 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 150$, which gives the height (in meters) of an object that has fallen from a height of 150 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but $\lim_{x \to 0} [f(x) + g(x)]$ does exist.

106. Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

107. Prove Property 1 of Theorem 1.1.

108. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Prove Property 1 of Theorem 1.2.

110. Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

111. Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x \to c} f(x)g(x) = 0$.

112. (a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$. (*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$. [*Hint:* Use the inequality $\|f(x)| - |L\| \le |f(x) - L|$.]

True or False? **In Exercises 113–118, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

113. $\lim_{x \to 0} \frac{|x|}{x} = 1$

114. $\lim_{x \to 0} x^3 = 0$

115. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and

$$\lim_{x \to 0} f(x) = L$$

then

$$\lim_{x \to 0} g(x) = L.$$

116. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

117. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

118. If $f(x) < g(x)$ for all $x \ne a$, then

$$\lim_{x \to a} f(x) < \lim_{x \to a} g(x).$$

119. ***Think About It*** Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

120. Prove the second part of Theorem 1.9 by proving that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

121. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

122. ***Graphical Reasoning*** Consider $f(x) = \frac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm the answer in part (c) analytically.

123. ***Approximation***

(a) Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

(b) Use the result in part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use the result in part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

124. ***Think About It*** When using a graphing utility to generate a table to approximate $\lim_{x \to 0} [(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a+b)/2]$ or $[(a+b)/2, b]$. From the sign of $f([a+b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

TECHNOLOGY You can also use the *zoom* feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.38.

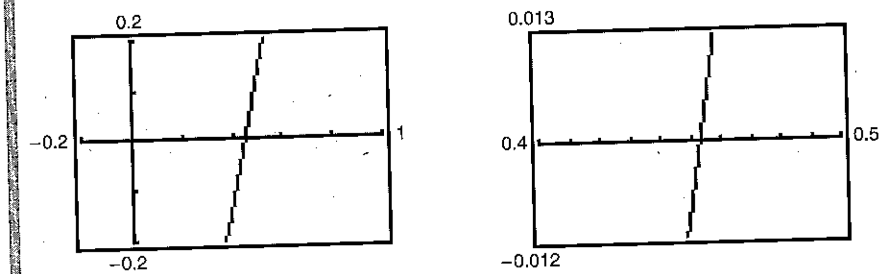


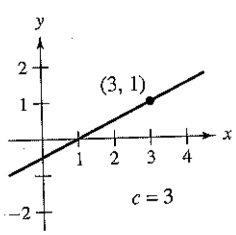
Figure 1.38 Zooming in on the zero of $f(x) = x^3 + 2x - 1$

EXERCISES FOR SECTION 1.4

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

1.



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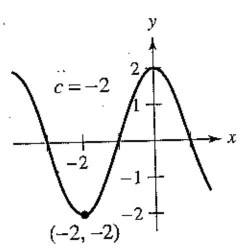

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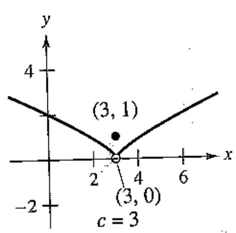

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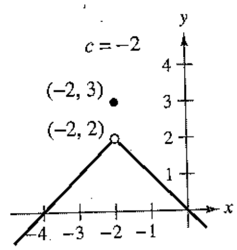
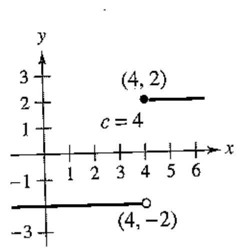
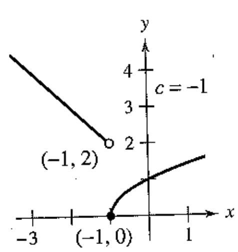


5.



6.



In Exercises 7–24, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x \to 5^+} \dfrac{x - 5}{x^2 - 25}$

8. $\lim_{x \to 2^+} \dfrac{2 - x}{x^2 - 4}$

9. $\lim_{x \to -3^-} \dfrac{x}{\sqrt{x^2 - 9}}$

10. $\lim_{x \to 4^-} \dfrac{\sqrt{x} - 2}{x - 4}$

11. $\lim_{x \to 0^-} \dfrac{|x|}{x}$

12. $\lim_{x \to 2^+} \dfrac{|x - 2|}{x - 2}$

13. $\lim_{\Delta x \to 0^-} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x}$

14. $\lim_{\Delta x \to 0^+} \dfrac{(x + \Delta x)^2 + x + \Delta x - (x^2 + x)}{\Delta x}$

15. $\lim_{x \to 3^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x + 2}{2}, & x \le 3 \\ \dfrac{12 - 2x}{3}, & x > 3 \end{cases}$

16. $\lim_{x \to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

17. $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

18. $\lim_{x \to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

19. $\lim_{x \to \pi} \cot x$

20. $\lim_{x \to \pi/2} \sec x$

21. $\lim_{x \to 4^-} (3[\![x]\!] - 5)$

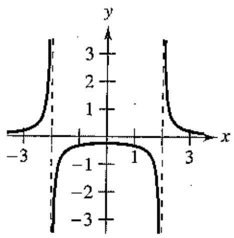
22. $\lim_{x \to 2^+} (2x - [\![x]\!])$

23. $\lim_{x \to 3} (2 - [\![-x]\!])$

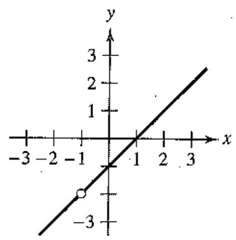
24. $\lim_{x \to 1} \left(1 - \left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

In Exercises 25–28, discuss the continuity of each function.

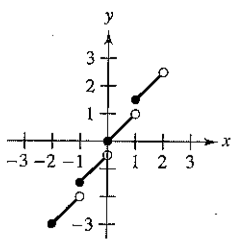
25. $f(x) = \dfrac{1}{x^2 - 4}$



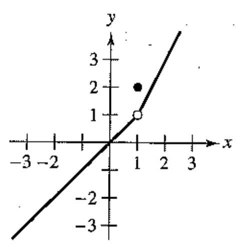
26. $f(x) = \dfrac{x^2 - 1}{x + 1}$



27. $f(x) = \frac{1}{2}[\![x]\!] + x$



28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



In Exercises 29–32, discuss the continuity of the function on the closed interval.

29. $g(x) = \sqrt{25 - x^2}$, $[-5, 5]$

30. $f(t) = 3 - \sqrt{9 - t^2}$, $[-3, 3]$

31. $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{1}{2}x, & x > 0 \end{cases}$, $[-1, 4]$

32. $g(x) = \dfrac{1}{x^2 - 4}$, $[-1, 2]$

In Exercises 33–54, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

33. $f(x) = x^2 - 2x + 1$

34. $f(x) = \dfrac{1}{x^2 + 1}$

35. $f(x) = 3x - \cos x$

36. $f(x) = \cos \dfrac{\pi x}{2}$

37. $f(x) = \dfrac{x}{x^2 - x}$

38. $f(x) = \dfrac{x}{x^2 - 1}$

39. $f(x) = \dfrac{x}{x^2 + 1}$

40. $f(x) = \dfrac{x - 3}{x^2 - 9}$

41. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

42. $f(x) = \dfrac{x - 1}{x^2 + x - 2}$

43. $f(x) = \dfrac{|x + 2|}{x + 2}$

44. $f(x) = \dfrac{|x - 3|}{x - 3}$

45. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

47. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

48. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

49. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

50. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \dfrac{\pi x}{2}$

53. $f(x) = [\![x - 1]\!]$

54. $f(x) = 3 - [\![x]\!]$

In Exercises 55 and 56, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x \to 0^+} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

55. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

56. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 57–60, find the constants a and b such that the function is continuous on the entire real line.

57. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

58. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

59. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

60. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 61–64, discuss the continuity of the composite function $h(x) = f(g(x))$.

61. $f(x) = x^2$
$g(x) = x - 1$

62. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

63. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

64. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 65–68, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

65. $f(x) = [\![x]\!] - x$

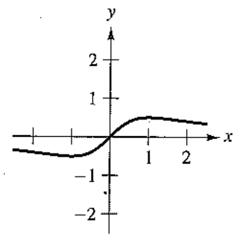
66. $h(x) = \dfrac{1}{x^2 - x - 2}$

67. $g(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

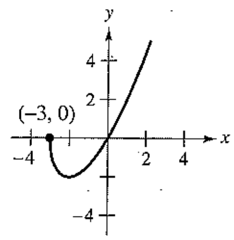
68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 69–72, describe the interval(s) on which the function is continuous.

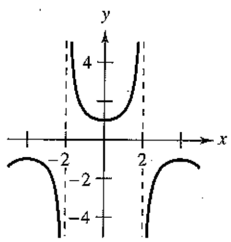
69. $f(x) = \dfrac{x}{x^2 + 1}$



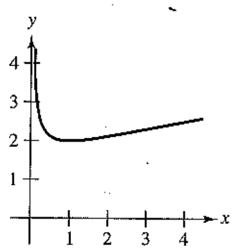
70. $f(x) = x\sqrt{x+3}$



71. $f(x) = \sec \dfrac{\pi x}{4}$



72. $f(x) = \dfrac{x+1}{\sqrt{x}}$



Writing **In Exercises 73 and 74, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

73. $f(x) = \dfrac{\sin x}{x}$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 75–78, explain why the function has a zero in the specified interval.**

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3, \quad [1, 2]$

76. $f(x) = x^3 + 3x - 2, \quad [0, 1]$

77. $f(x) = x^2 - 2 - \cos x, \quad [0, \pi]$

78. $f(x) = -\dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right), \quad [1, 3]$

In Exercises 79–82, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval $[0, 1]$. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the root-finding capabilities of the graphing utility to approximate the zero accurate to four decimal places.

79. $f(x) = x^3 + x - 1$

80. $f(x) = x^3 + 3x - 2$

81. $g(t) = 2\cos t - 3t$

82. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 83–86, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

83. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

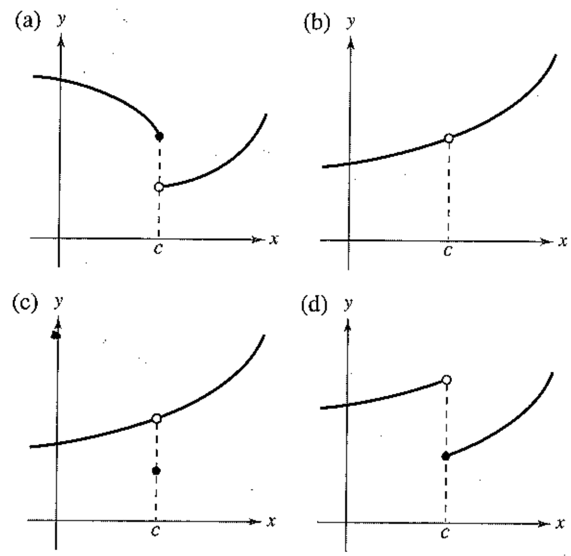
84. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

85. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

86. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

Getting at the Concept

87. State how continuity is destroyed at $x = c$ for each of the following.



88. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following.
   (a) A function with a nonremovable discontinuity at $x = 2$.
   (b) A function with a removable discontinuity at $x = -2$.
   (c) A function that has both of the characteristics described in parts (a) and (b).

89. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

90. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

91. ***Think About It*** Describe how the functions $f(x) = 3 + [\![x]\!]$ and $g(x) = 3 - [\![-x]\!]$ differ.

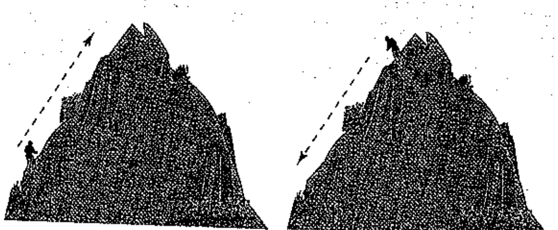
92. ***Telephone Charges*** A dial-direct long distance call between two cities costs \$1.04 for the first 2 minutes and \$0.36 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of the time t (in minutes). Sketch a graph of this function and discuss its continuity.

93. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

94. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Saturday 8:00 A.M. Sunday 8:00 A.M. *Not drawn to scale*

95. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[1, 5]$, there is one with a volume of 275 cubic centimeters.

96. Prove that if f is continuous and has no zeros on $[a, b]$, then either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

97. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

98. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

99. The **signum function** is defined by

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\text{sgn}(x)$ and find the following (if possible).

(a) $\lim_{x\to0^-} \text{sgn}(x)$ (b) $\lim_{x\to0^+} \text{sgn}(x)$ (c) $\lim_{x\to0} \text{sgn}(x)$

True or False? **In Exercises 100–103, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

100. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

101. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

102. A rational function can have infinitely many x-values at which it is not continuous.

103. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

104. ***Modeling Data*** After an object falls for t seconds, the speed S (in feet per second) of the object is recorded in the table.

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

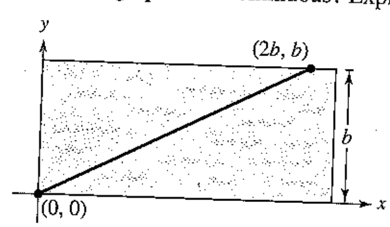
(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

105. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)

(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's path across the pool. Determine the function f and sketch its graph. Is it continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is it continuous? Explain.



106. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

107. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

108. Prove that if $\lim_{\Delta x\to0} f(c + \Delta x) = f(c)$, then f is continuous at c.

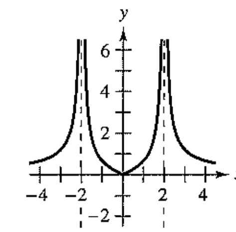
109. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

110. Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

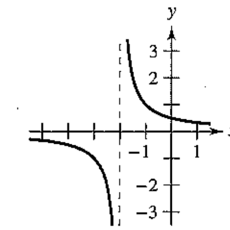
EXERCISES FOR SECTION 1.5

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

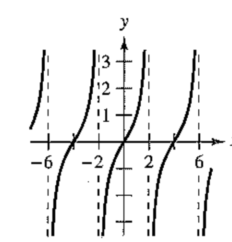
1. $f(x) = 2\left|\dfrac{x}{x^2-4}\right|$



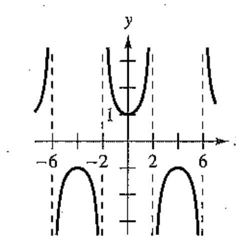
2. $f(x) = \dfrac{1}{x+2}$



3. $f(x) = \tan\dfrac{\pi x}{4}$



4. $f(x) = \sec\dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function and confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

5. $f(x) = \dfrac{1}{x^2-9}$

6. $f(x) = \dfrac{x}{x^2-9}$

7. $f(x) = \dfrac{x^2}{x^2-9}$

8. $f(x) = \sec\dfrac{\pi x}{6}$

In Exercises 9–28, find the vertical asymptotes (if any) of the function.

9. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = \dfrac{4}{(x-2)^3}$

11. $h(x) = \dfrac{x^2-2}{x^2-x-2}$

12. $g(x) = \dfrac{2+x}{x^2(1-x)}$

13. $f(x) = \dfrac{x^2}{x^2-4}$

14. $f(x) = \dfrac{-4x}{x^2+4}$

15. $g(t) = \dfrac{t-1}{t^2+1}$

16. $h(s) = \dfrac{2s-3}{s^2-25}$

17. $f(x) = \tan 2x$

18. $f(x) = \sec \pi x$

19. $T(t) = 1 - \dfrac{4}{t^2}$

20. $g(x) = \dfrac{\frac{1}{2}x^3 - x^2 - 4x}{3x^2 - 6x - 24}$

21. $f(x) = \dfrac{x}{x^2+x-2}$

22. $f(x) = \dfrac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$

23. $g(x) = \dfrac{x^3+1}{x+1}$

24. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2}$

25. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

26. $h(t) = \dfrac{t^2-2t}{t^4-16}$

27. $s(t) = \dfrac{t}{\sin t}$

28. $g(\theta) = \dfrac{\tan\theta}{\theta}$

In Exercises 29–32, determine whether the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

29. $f(x) = \dfrac{x^2-1}{x+1}$

30. $f(x) = \dfrac{x^2-6x-7}{x+1}$

31. $f(x) = \dfrac{x^2+1}{x+1}$

32. $f(x) = \dfrac{\sin(x+1)}{x+1}$

In Exercises 33–48, find the limit.

33. $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x-2}$

34. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$

35. $\displaystyle\lim_{x\to 3^+} \frac{x^2}{x^2-9}$

36. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

37. $\displaystyle\lim_{x\to -3^-} \frac{x^2+2x-3}{x^2+x-6}$

38. $\displaystyle\lim_{x\to (-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$

39. $\displaystyle\lim_{x\to 1} \frac{x^2-x}{(x^2+1)(x-1)}$

40. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$

41. $\displaystyle\lim_{x\to 0^-} \left(1+\frac{1}{x}\right)$

42. $\displaystyle\lim_{x\to 0^-} \left(x^2-\frac{1}{x}\right)$

43. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$

44. $\displaystyle\lim_{x\to (\pi/2)^+} \frac{-2}{\cos x}$

45. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x}$

46. $\displaystyle\lim_{x\to 0} \frac{x+2}{\cot x}$

47. $\displaystyle\lim_{x\to 1/2} x\sec \pi x$

48. $\displaystyle\lim_{x\to 1/2} x^2\tan \pi x$

In Exercises 49–52, use a graphing utility to graph the function and determine the one-sided limit.

49. $f(x) = \dfrac{x^2+x+1}{x^3-1}$

$\displaystyle\lim_{x\to 1^+} f(x)$

50. $f(x) = \dfrac{x^3-1}{x^2+x+1}$

$\displaystyle\lim_{x\to 1^-} f(x)$

51. $f(x) = \dfrac{1}{x^2-25}$

$\displaystyle\lim_{x\to 5^-} f(x)$

52. $f(x) = \sec\dfrac{\pi x}{6}$

$\displaystyle\lim_{x\to 3^+} f(x)$

Getting at the Concept

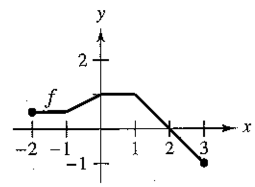
53. In your own words, describe the meaning of an infinite limit. Is ∞ a real number?

54. In your own words, describe what is meant by an asymptote of a graph.

55. Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Does every rational function have a vertical asymptote? Explain.

57. Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



58. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

59. A given sum S is inversely proportional to $1 - r$, where $0 < |r| < 1$. Find the limit of S as $r \to 1^-$.

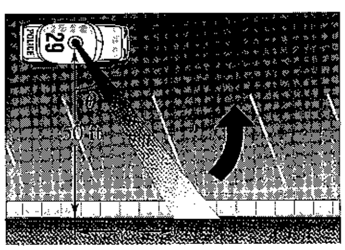
60. ***Rate of Change*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is

$$r = 50\pi \sec^2 \theta \text{ ft/sec}.$$

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.



61. ***Illegal Drugs*** The cost in millions of dollars for a governmental agency to seize $x\%$ of an illegal drug is

$$C = \frac{528x}{100 - x}, \quad 0 \le x < 100.$$

(a) Find the cost of seizing 25% of the drug.

(b) Find the cost of seizing 50% of the drug.

(c) Find the cost of seizing 75% of the drug.

(d) Find the limit of C as $x \to 100^-$ and interpret its meaning.

62. ***Relativity*** According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

63. ***Rate of Change*** A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

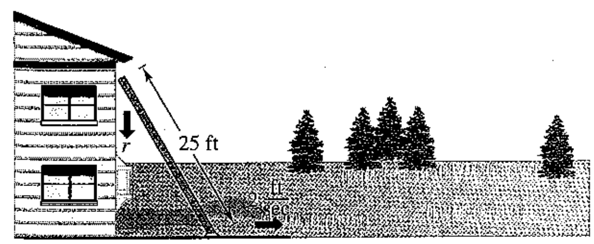
$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.



64. ***Average Speed*** On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as $x \to 25^+$ and interpret its meaning.

65. ***Numerical and Graphical Analysis*** Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power on x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

(a) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x}$

(b) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^2}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^3}$

(d) $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^4}$

66. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).

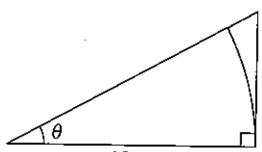
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Use a graphing utility to graph the function over the appropriate domain.

(d) Find the limit of A as $\theta \to (\pi/2)^-$.



67. ***Numerical and Graphical Reasoning*** A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.

(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.)

(d) Use a graphing utility to complete the table.

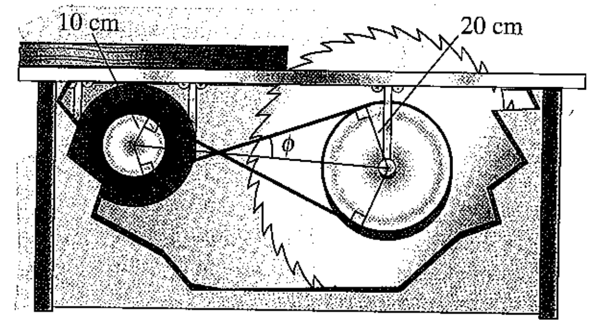
ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find $\lim_{\phi \to (\pi/2)^-} L$.

Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.



True or False? **In Exercises 68–71, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

68. If $p(x)$ is a polynomial, then the function given by

$$f(x) = \frac{p(x)}{x - 1}$$

has a vertical asymptote at $x = 1$.

69. A rational function has at least one vertical asymptote.

70. Polynomial functions have no vertical asymptotes.

71. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

72. Find functions f and g such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = \infty$$

but $\lim_{x \to c} [f(x) - g(x)] \neq 0$.

73. Prove the remaining properties of Theorem 1.15.

74. Prove that if $\lim_{x \to c} f(x) = \infty$ then $\lim_{x \to c} \frac{1}{f(x)} = 0$.

75. Prove that if $\lim_{x \to c} \frac{1}{f(x)} = 0$ then $\lim_{x \to c} f(x)$ does not exist.

SECTION PROJECT GRAPHS AND LIMITS OF TRIGONOMETRIC FUNCTIONS

Recall from Theorem 1.9 that the limit of $f(x) = (\sin x)/x$ as x approaches 0 is 1.

(a) Use a graphing utility to graph the function f on the interval $-\pi \le 0 \le \pi$. Explain how this graph helps confirm that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula for $x = 0.1$ and $x = 0.01$. Then find the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically.

(f) Find the slope of the tangent line to $k(x) = \tan x$ at $(0, 0)$.