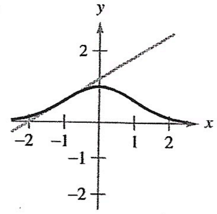
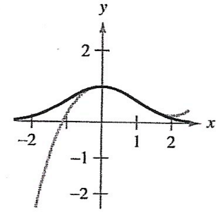
EXERCISES FOR SECTION 8.7

In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

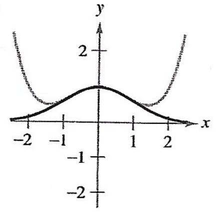
(a)



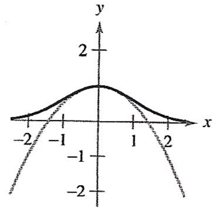
(b)



(c)



(d)



1. $g(x) = -\frac{1}{2}x^2 + 1$
2. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$
3. $g(x) = e^{-1/2}[(x + 1) + 1]$
4. $g(x) = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

In Exercises 5–8, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1. What is P_1 called?

5. $f(x) = \dfrac{4}{\sqrt{x}}$, $c = 1$
6. $f(x) = \dfrac{4}{\sqrt[3]{x}}$, $c = 8$
7. $f(x) = \sec x$, $c = \dfrac{\pi}{4}$
8. $f(x) = \tan x$, $c = \dfrac{\pi}{4}$

Graphical and Numerical Analysis **In Exercises 9 and 10, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2.**

9. $f(x) = \dfrac{4}{\sqrt{x}}$, $c = 1$

$$P_2(x) = 4 - 2(x - 1) + \tfrac{3}{2}(x - 1)^2$$

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$							
$P_2(x)$							

10. $f(x) = \sec x$, $c = \dfrac{\pi}{4}$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

x	−2.15	0.585	0.685	$\frac{\pi}{4}$	0.885	0.985	1.785
$f(x)$							
$P_2(x)$							

11. ***Conjecture*** Consider the function $f(x) = \cos x$ and its Maclaurin polynomials P_2, P_4, and P_6 (see Example 5).
   (a) Use a graphing utility to graph f and the indicated polynomial approximations.
   (b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 4$, and 6.
   (c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

12. ***Conjecture*** Consider the function $f(x) = x^2e^x$.
   (a) Find the Maclaurin polynomials P_2, P_3, and P_4 for f.
   (b) Use a graphing utility to graph f, P_2, P_3, and P_4.
   (c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 3$, and 4.
   (d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

In Exercises 13–24, find the Maclaurin polynomial of degree n for the function.

13. $f(x) = e^{-x}$, $n = 3$
14. $f(x) = e^{-x}$, $n = 5$
15. $f(x) = e^{2x}$, $n = 4$
16. $f(x) = e^{3x}$, $n = 4$
17. $f(x) = \sin x$, $n = 5$
18. $f(x) = \sin \pi x$, $n = 3$
19. $f(x) = xe^x$, $n = 4$
20. $f(x) = x^2e^{-x}$, $n = 4$
21. $f(x) = \dfrac{1}{x + 1}$, $n = 4$
22. $f(x) = \dfrac{x}{x + 1}$, $n = 4$
23. $f(x) = \sec x$, $n = 2$
24. $f(x) = \tan x$, $n = 3$

In Exercises 25–30, find the nth Taylor polynomial centered at c.

25. $f(x) = \dfrac{1}{x}$, $n = 4$, $c = 1$
26. $f(x) = \dfrac{2}{x^2}$, $n = 4$, $c = 2$
27. $f(x) = \sqrt{x}$, $n = 4$, $c = 1$
28. $f(x) = \sqrt[3]{x}$, $n = 3$, $c = 8$
29. $f(x) = \ln x$, $n = 4$, $c = 1$
30. $f(x) = x^2 \cos x$, $n = 2$, $c = \pi$

In Exercises 53 and 54, determine the values of x for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.

53. $f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \quad x < 0$

54. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

Getting at the Concept

55. An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c*.

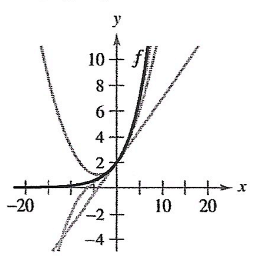
56. When an elementary function f is approximated by a second-degree polynomial P_2 centered at c, what is known about f and P_2 at c?

57. State the definition of an nth-degree Taylor polynomial of f centered at c.

58. Describe the accuracy of the nth-degree Taylor polynomial of f centered at c as the distance between c and x increases.

59. In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial is increased?

60. The graphs show first-, second-, and third-degree polynomial approximations P_1, P_2, and P_3 of a function f. Label the graphs of P_1, P_2, and P_3. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



61. ***Comparing Maclaurin Polynomials***
    (a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions

    $f(x) = e^x \quad$ and $\quad g(x) = xe^x.$

    What is the relationship between them?
    (b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.
    (c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.

62. ***Differentiating Maclaurin Polynomials***
    (a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.
    (b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.
    (c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

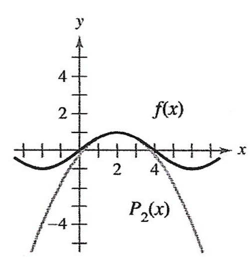
63. ***Graphical Reasoning*** The figure shows the graph of the function

    $$f(x) = \sin\left(\frac{\pi x}{4}\right)$$

    and the second-degree Taylor polynomial

    $$P_2(x) = 1 - \frac{\pi^2}{32}(x - 2)^2$$

    centered at $x = 2$.
    (a) Use the symmetry of the graph of f to write the second-degree Taylor polynomial for f centered at $x = -2$.
    (b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial for f centered at $x = 6$.
    (c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for f centered at $x = 4$? Explain.



64. Prove that if f is an odd function, then its nth Maclaurin polynomial contains only terms with odd powers of x.

65. Prove that if f is an even function, then its nth Maclaurin polynomial contains only terms with even powers of x.

66. Let $P_n(x)$ be the nth Taylor polynomial for f at c. Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \le k \le n$. (See Exercises 9 and 10.)

67. ***Writing*** The proof in Exercise 66 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.

In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function f. Graph the function and the Taylor polynomials.

31. $f(x) = \tan x$
 (a) $n = 3, \quad c = 0$
 (b) $n = 5, \quad c = 0$
 (c) $n = 3, \quad c = \pi/4$

32. $f(x) = \dfrac{1}{x^2 + 1}$
 (a) $n = 2, \quad c = 0$
 (b) $n = 4, \quad c = 0$
 (c) $n = 4, \quad c = 1$

33. ***Numerical and Graphical Approximations***
 (a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ for $f(x) = \sin x$ to complete the table.

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$					
$P_3(x)$					
$P_5(x)$					
$P_7(x)$					

 (b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).
 (c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

34. ***Numerical and Graphical Approximations***
 (a) Use the Taylor polynomials $P_1(x)$ and $P_4(x)$ for $f(x) = \ln x$ centered at $c = 1$ to complete the table.

x	1.00	1.25	1.50	1.75	2.00
$\ln x$	0	0.2231	0.4055	0.5596	0.6931
$P_1(x)$					
$P_4(x)$					

 (b) Use a graphing utility to graph $f(x) = \ln x$ and the Taylor polynomials in part (a).
 (c) Describe the change in accuracy of polynomial approximations as the degree increases.

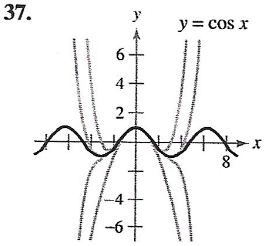
Numerical and Graphical Approximations **In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.**

x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$							
$P_3(x)$							

35. $f(x) = \arcsin x$

36. $f(x) = \arctan x$

In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.

37.


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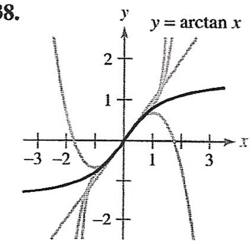

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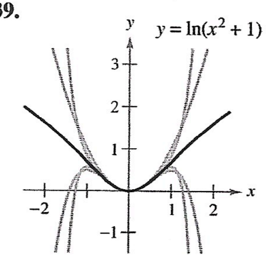

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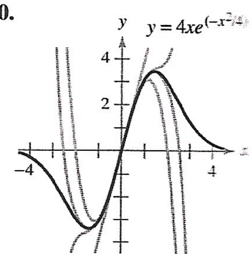

In Exercises 41–44, approximate the function at the given va of x, using the polynomial found in the indicated exercise.

41. $f(x) = e^{-x}$, $\quad f(\frac{1}{2})$, Exercise 13
42. $f(x) = x^2e^{-x}$, $\quad f(\frac{1}{5})$, Exercise 20
43. $f(x) = \ln x$, $\quad f(1.2)$, Exercise 29
44. $f(x) = x^2 \cos x$, $\quad f\left(\dfrac{7\pi}{8}\right)$, Exercise 30

In Exercises 45–48, use Taylor's Theorem to obtain an upp bound for the error of the approximation. Then calculate th exact value of the error.

45. $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

46. $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

47. $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2 \cdot 3}$

48. $\arctan(0.4) \approx 0.4 -$ [illegible]

In Exercises 49 and 50, determine the degree of the Mac polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.

49. $\sin(0.3)$

50. $e^{0.6}$

In Exercises 51 and 52, determine the degree of the Mac polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.0001. Use computer algebra system to obtain and evaluate the req derivatives.

51. $f(x) = \ln(x + 1)$, approximate $f(0.5)$.
52. $f(x) = \cos(\pi x^2)$, approximate $f(0.6)$.