EXERCISES FOR SECTION 3.8

In Exercises 1–4, complete two iterations of Newton's Method for the function using the indicated initial guess.

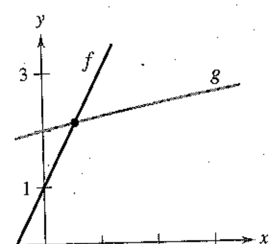
1. $f(x) = x^2 - 3, \quad x_1 = 1.7$
2. $f(x) = 2x^2 - 3, \quad x_1 = 1$
3. $f(x) = \sin x, \quad x_1 = 3$
4. $f(x) = \tan x, \quad x_1 = 0.1$

In Exercises 5–14, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

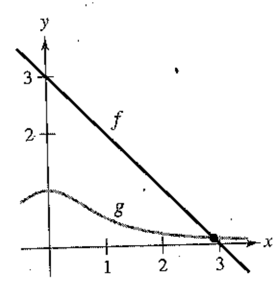
5. $f(x) = x^3 + x - 1$
6. $f(x) = x^5 + x - 1$
7. $f(x) = 3\sqrt{x-1} - x$
8. $f(x) = x - 2\sqrt{x+1}$
9. $f(x) = x^3 + 3$
10. $f(x) = 1 - 2x^3$
11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$
12. $f(x) = \frac{1}{2}x^4 - 3x - 3$
13. $f(x) = x + \sin(x + 1)$
14. $f(x) = x^3 - \cos x$

In Exercises 15–18, apply Newton's Method to approximate the x-value of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]

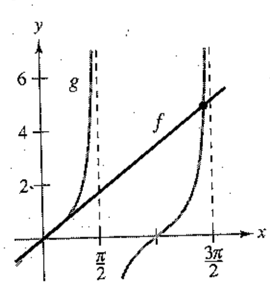
15. $f(x) = 2x + 1$
$g(x) = \sqrt{x + 4}$



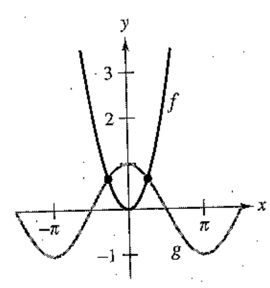
16. $f(x) = 3 - x$
$g(x) = 1/(x^2 + 1)$



17. $f(x) = x$
$g(x) = \tan x$



18. $f(x) = x^2$
$g(x) = \cos x$



In Exercises 19 and 20, use Newton's Method to obtain a general rule for approximating the required radical.

19. $x = \sqrt{a}$ [*Hint:* Consider $f(x) = x^2 - a$.]
20. $x = \sqrt[n]{a}$ [*Hint:* Consider $f(x) = x^n - a$.]

In Exercises 21–24, use the results of Exercises 19 and 20 to approximate the indicated radical to three decimal places.

21. $\sqrt{7}$
22. $\sqrt{5}$
23. $\sqrt[4]{6}$
24. $\sqrt[3]{15}$

In Exercises 25 and 26, approximate π to three decimal places using Newton's Method and the given function.

25. $f(x) = 1 + \cos x$
26. $f(x) = \tan x$

In Exercises 27–30, apply Newton's Method using the indicated initial guess, and explain why the method fails.

27. $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$
28. $y = 4x^3 - 12x^2 + 12x - 3, \quad x_1 = \frac{3}{2}$

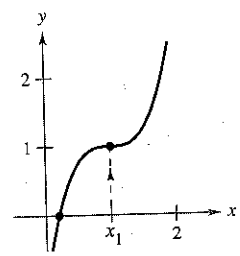


Figure for 27

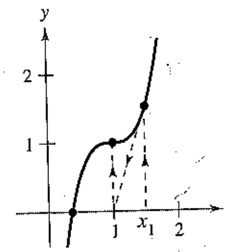


Figure for 28

29. $f(x) = -x^3 + 6x^2 - 10x + 6, \quad x_1 = 2$
30. $f(x) = 2\sin x + \cos 2x, \quad x_1 = \dfrac{3\pi}{2}$

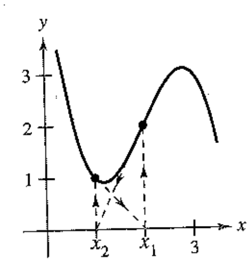


Figure for 29

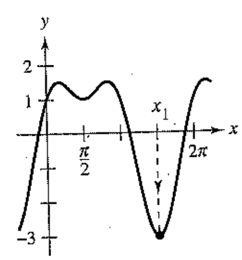


Figure for 30

Getting at the Concept

31. In your own words and using a sketch, describe Newton's Method for approximating the zeros of a function.
32. Under what conditions will Newton's Method fail?

Fixed Point **In Exercises 33 and 34, approximate the fixed point of the function to two decimal places. [A *fixed point* x_0 of a function f is a value of x such that $f(x_0) = x_0$.]**

33. $f(x) = \cos x$
34. $f(x) = \cot x, \quad 0 < x < \pi$

35. ***Writing*** Consider the function $f(x) = x^3 - 3x^2 + 3$.

(a) Use a graphing utility to obtain the graph of f.

(b) Use Newton's Method with $x_1 = 1$ as an initial guess.

(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.

(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.

(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

36. ***Writing*** Repeat the steps in Exercise 35 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

37. Use Newton's Method to show that the equation

$$x_{n+1} = x_n(2 - ax_n)$$

can be used to approximate $1/a$ if x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

38. Use the result of Exercise 37 to approximate the indicated reciprocal to three decimal places.

(a) $\frac{1}{3}$ (b) $\frac{1}{11}$

In Exercises 39 and 40, approximate the critical number of f on the interval $(0, \pi)$. Sketch the graph of f, labeling any extrema.

39. $f(x) = x \cos x$

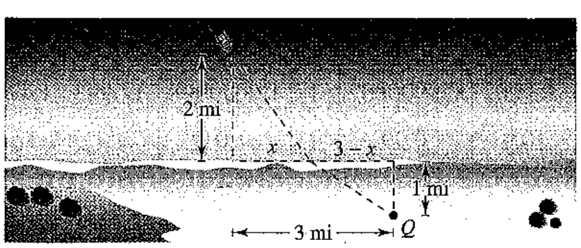
40. $f(x) = x \sin x$

In Exercises 41–44, we review some typical problems from the previous sections of this chapter. In each case, use Newton's Method to approximate the solution.

41. ***Minimum Distance*** Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

42. ***Minimum Distance*** Find the point on the graph of $f(x) = x^2$ that is closest to the point $(4, -3)$.

43. ***Minimum Time*** You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point Q, which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach Q in the least time?



44. ***Medicine*** The concentration C of a certain chemical in the bloodstream t hours after injection into muscle tissue is given by $C = (3t^2 + t)/(50 + t^3)$. When is the concentration greatest?

45. ***Advertising Costs*** A company that produces portable cassette players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where P is the profit in dollars and x is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of two advertising amounts that yield a profit P of \$2,500,000.

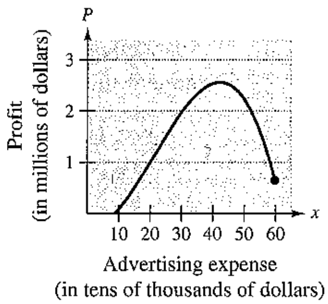


Figure for 45

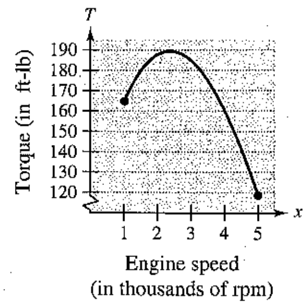


Figure for 46

46. ***Engine Power*** The torque produced by a compact automobile engine is approximated by the model

$$T = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, \quad 1 \le x \le 5$$

where T is the torque in foot-pounds and x is the engine speed in thousands of revolutions per minute (see figure). Approximate the two engine speeds that yield a torque T of 170 foot-pounds.

True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. The zeros of $f(x) = p(x)/q(x)$ coincide with the zeros of $p(x)$.

48. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

49. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

50. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

In Exercises 51 and 52, ***write a computer program or use a spreadsheet*** **to find the zeros of a function using Newton's Method. Approximate the zeros of the function accurate to three decimal places. The output should be a table with the following headings.**

$$n, \; x_n, \; f(x_n), \; f'(x_n), \; \frac{f(x_n)}{f'(x_n)}, \; x_n - \frac{f(x_n)}{f'(x_n)}$$

51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

52. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

EXERCISES FOR SECTION 5.1

1. Complete the table below. Use a graphing utility and Simpson's Rule with $n = 10$ to approximate the integral

$$\int_1^x \frac{1}{t}\,dt.$$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x (1/t)\,dt$							

2. (a) Plot the points generated in Exercise 1 and connect them with a smooth curve. Compare the result with the graph of $y = \ln x$.
   (b) Use a graphing utility to graph $y = \int_1^x (1/t)\,dt$ for $0.2 \le x \le 4$. Compare the result with the graph of $y = \ln x$.

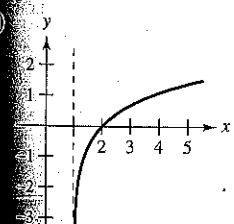
In Exercises 3–6, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key, and (b) using the integration capabilities to evaluate the integral

$$\int_1^x \frac{1}{t}\,dt.$$

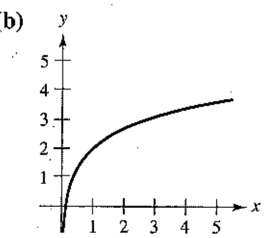
3. $\ln 45$
4. $\ln 8.3$
5. $\ln 0.8$
6. $\ln 0.6$

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

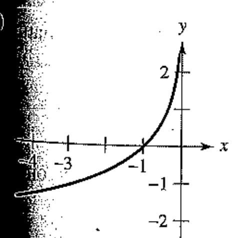
(a)



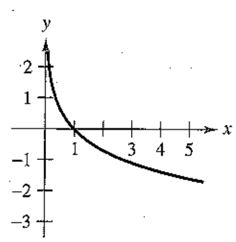
(b)



(c)



(d)



7. $f(x) = \ln x + 2$
8. $f(x) = -\ln x$
9. $f(x) = \ln(x - 1)$
10. $f(x) = -\ln(-x)$

In Exercises 11–16, sketch the graph of the function and state its domain.

11. $f(x) = 3 \ln x$
12. $f(x) = -2 \ln x$
13. $f(x) = \ln 2x$
14. $f(x) = \ln|x|$
15. $f(x) = \ln(x - 1)$
16. $g(x) = 2 + \ln x$

In Exercises 17 and 18, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

17. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$
18. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 19–28, use the properties of logarithms to expand the logarithmic expression.

19. $\ln \frac{2}{3}$
20. $\ln \sqrt{2^3}$
21. $\ln \dfrac{xy}{z}$
22. $\ln(xyz)$
23. $\ln \sqrt[3]{a^2 + 1}$
24. $\ln \sqrt{a - 1}$
25. $\ln\left(\dfrac{x^2 - 1}{x^3}\right)^3$
26. $\ln(3e^2)$
27. $\ln z(z - 1)^2$
28. $\ln \dfrac{1}{e}$

In Exercises 29–34, write the expression as a logarithm of a single quantity.

29. $\ln(x - 2) - \ln(x + 2)$
30. $3 \ln x + 2 \ln y - 4 \ln z$
31. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$
32. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$
33. $2 \ln 3 - \frac{1}{2} \ln(x^2 + 1)$
34. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

In Exercises 35 and 36, show that $f = g$ by using a graphing utility to graph f and g in the same viewing window.

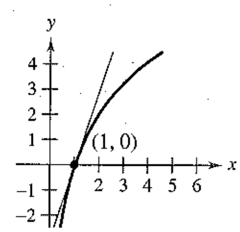
35. $f(x) = \ln \dfrac{x^2}{4},\ x > 0, \quad g(x) = 2 \ln x - \ln 4$
36. $f(x) = \ln\sqrt{x(x^2 + 1)}, \quad g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

In Exercises 37–40, find the limit.

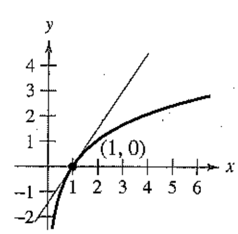
37. $\lim_{x\to 3^+} \ln(x - 3)$
38. $\lim_{x\to 6^-} \ln(6 - x)$
39. $\lim_{x\to 2^-} \ln[x^2(3 - x)]$
40. $\lim_{x\to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

In Exercises 41–44, find the slope of the tangent line to the graph of the logarithmic function at the point (1, 0).

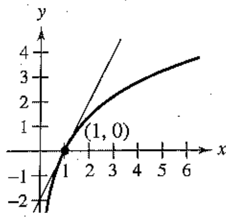
41. $y = \ln x^3$



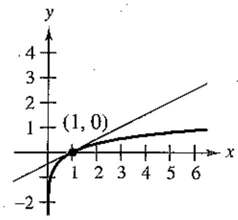
42. $y = \ln x^{3/2}$



43. $y = \ln x^2$



44. $y = \ln x^{1/2}$



In Exercises 45–70, find the derivative of the function.

45. $g(x) = \ln x^2$

46. $h(x) = \ln(2x^2 + 1)$

47. $y = (\ln x)^4$

48. $y = x \ln x$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

50. $y = \ln\sqrt{x^2 - 4}$

51. $f(x) = \ln\left(\dfrac{x}{x^2 + 1}\right)$

52. $f(x) = \ln\left(\dfrac{2x}{x + 3}\right)$

53. $g(t) = \dfrac{\ln t}{t^2}$

54. $h(t) = \dfrac{\ln t}{t}$

55. $y = \ln(\ln x^2)$

56. $y = \ln(\ln x)$

57. $y = \ln\sqrt{\dfrac{x + 1}{x - 1}}$

58. $y = \ln\sqrt[3]{\dfrac{x - 1}{x + 1}}$

59. $f(x) = \ln\left(\dfrac{\sqrt{4 + x^2}}{x}\right)$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

61. $y = \dfrac{-\sqrt{x^2 + 1}}{x} + \ln\left(x + \sqrt{x^2 + 1}\right)$

62. $y = \dfrac{-\sqrt{x^2 + 4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2 + \sqrt{x^2 + 4}}{x}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

67. $y = \ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right|$

68. $y = \ln\sqrt{1 + \sin^2 x}$

69. $f(x) = \sin 2x \ln x^2$

70. $g(x) = \displaystyle\int_1^{\ln x} (t^2 + 3)\, dt$

In Exercises 71 and 72, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
71.	$f(x) = 3x^2 - \ln x$	$(1, 3)$
72.	$f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right)$	$(0, 4)$

In Exercises 73 and 74, use implicit differentiation to find dy/dx.

73. $x^2 - 3\ln y + y^2 = 10$

74. $\ln xy + 5x = 30$

In Exercises 75 and 76, show that the function is a solution of the differential equation.

	Function	*Differential Equation*
75.	$y = 2\ln x + 3$	$xy'' + y' = 0$
76.	$y = x\ln x - 4x$	$x + y - xy' = 0$

In Exercises 77–82, locate any relative extrema and inflection points. Use a graphing utility to confirm your results.

77. $y = \dfrac{x^2}{2} - \ln x$

78. $y = x - \ln x$

79. $y = x \ln x$

80. $y = \dfrac{\ln x}{x}$

81. $y = \dfrac{x}{\ln x}$

82. $y = x^2 \ln \dfrac{x}{4}$

Linear and Quadratic Approximations **In Exercises 83 and 84, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(1) + f'(1)(x - 1)$$

and

$$P_2(x) = f(1) + f'(1)(x - 1) + \tfrac{1}{2}f''(1)(x - 1)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 1$.

83. $f(x) = \ln x$

84. $f(x) = x \ln x$

In Exercises 85 and 86, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.

85. $y = \ln x$
$y = -x$

86. $y = \ln x$
$y = 3 - x$

In Exercises 87–92, find dy/dx using logarithmic differentiation.

87. $y = x\sqrt{x^2 - 1}$

88. $y = \sqrt{(x - 1)(x - 2)(x - 3)}$

89. $y = \dfrac{x^2\sqrt{3x - 2}}{(x - 1)^2}$

90. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

91. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$

92. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Getting at the Concept

93. In your own words, state the properties of the natural logarithmic function.

94. Define the base for the natural logarithmic function.

95. Explain why $\ln e^x = x$.

96. Let f be a function that is positive and differentiable on the entire real line. Let $g(x) = \ln f(x)$.

(a) If g is increasing, must f be increasing? Explain.

(b) If the graph of f is concave upward, must the graph of g be concave upward? Explain.

97. Consider the function $f(x) = x - 2\ln x$ on $[1, 3]$.

(a) Explain why Rolle's Theorem (Section 3.2) does not apply.

(b) Do you think the conclusion of Rolle's Theorem is true for f? Explain.

98. ***Home Mortgage*** The term t (in years) of a \$120,000 home mortgage at 10% interest can be approximated by

$$t = \frac{5.315}{-6.7968 + \ln x}, \quad x > 1000$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1167.41. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1068.45. What is the total amount paid?

(d) Find the instantaneous rate of change of t with respect to x when $x = 1167.41$ and $x = 1068.45$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

99. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-10} watts per square centimeter.

100. ***Modeling Data*** The table shows the temperature T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook of Mechanical Engineers)*

p	5	10	14.696 (1 atm)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rate of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find

$$\lim_{p\to\infty} T'(p)$$

and interpret the result in the context of the problem.

101. ***Modeling Data*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressure p (in atmospheres) at a given altitude h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the logarithmic model.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the model to estimate the pressure at an altitude of 13 kilometers.

(f) Find the rate of change of pressure when $h = 5$ and $h = 20$. Interpret the results in the context of the problem.

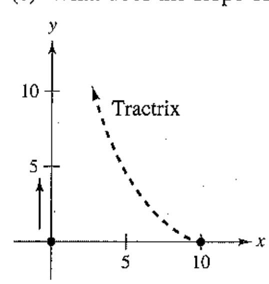
102. ***Tractrix*** A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What is the slope of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as $x \to 10$?



103. ***Conjecture*** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the faster rate for "large" values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x$, $\quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x$, $\quad g(x) = \sqrt[4]{x}$

104. Prove that the natural logarithmic function is one-to-one.

True or False? **In Exercises 105 and 106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. $\ln(x + 25) = \ln x + \ln 25$

106. If $y = \ln \pi$, then $y' = 1/\pi$.

EXERCISES FOR SECTION 5.4

In Exercises 1–4, write the exponential equation as a logarithmic equation or vice versa.

1. $e^0 = 1$
2. $e^{-2} = 0.1353 \ldots$
3. $\ln 2 = 0.6931 \ldots$
4. $\ln 0.5 = -0.6931 \ldots$

In Exercises 5–18, solve for x accurate to three decimal places.

5. $e^{\ln x} = 4$
6. $e^{\ln 2x} = 12$
7. $e^x = 12$
8. $4e^x = 83$
9. $9 - 2e^x = 7$
10. $-6 + 3e^x = 8$
11. $50e^{-x} = 30$
12. $200e^{-4x} = 15$
13. $\ln x = 2$
14. $\ln x^2 = 10$
15. $\ln(x - 3) = 2$
16. $\ln 4x = 1$
17. $\ln\sqrt{x + 2} = 1$
18. $\ln(x - 2)^2 = 12$

In Exercises 19–22, sketch the graph of the function.

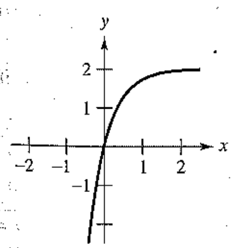
19. $y = e^{-x}$
20. $y = \frac{1}{2}e^x$
21. $y = e^{-x^2}$
22. $y = e^{-x/2}$

23. Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?
(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

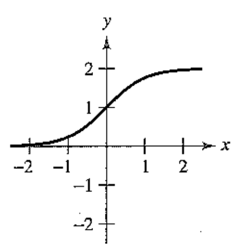
24. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.
(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 25–28, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]

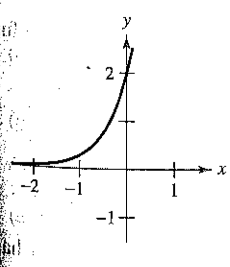
(a)



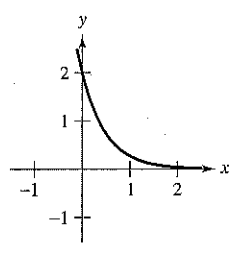
(b)



(c)



(d)



25. $y = Ce^{ax}$
26. $y = Ce^{-ax}$
27. $y = C(1 - e^{-ax})$
28. $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 29–32, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.

29. $f(x) = e^{2x}$
$g(x) = \ln\sqrt{x}$
30. $f(x) = e^{x/3}$
$g(x) = \ln x^3$
31. $f(x) = e^x - 1$
$g(x) = \ln(x + 1)$
32. $f(x) = e^{x-1}$
$g(x) = 1 + \ln x$

33. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as $x \to \infty$?

34. ***Conjecture*** Use the result of Exercise 33 to make a conjecture about the value of

$$\left(1 + \frac{r}{x}\right)^x$$

as $x \to \infty$.

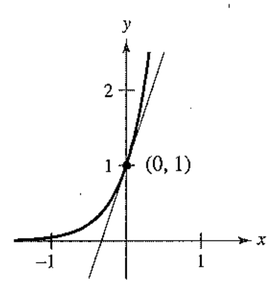
In Exercises 35 and 36, compare the given number with the number e. Is the number less than or greater than e?

35. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ (See Exercise 34.)

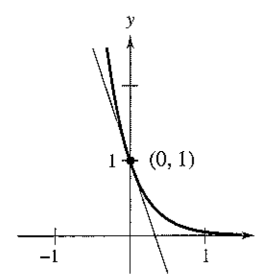
36. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040}$

In Exercises 37 and 38, find the slope of the tangent line to the graph of the function at the point (0, 1).

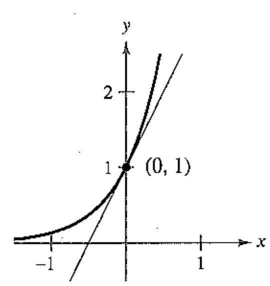
37. (a) $y = e^{3x}$



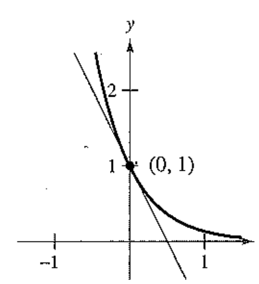
(b) $y = e^{-3x}$



38. (a) $y = e^{2x}$



(b) $y = e^{-2x}$



In Exercises 39–58, find the derivative of the function.

39. $f(x) = e^{2x}$
40. $f(x) = e^{1-x}$
41. $y = e^{-2x+x^2}$
42. $y = e^{-x^2}$
43. $y = e^{\sqrt{x}}$
44. $y = x^2e^{-x}$
45. $g(t) = (e^{-t} + e^{t})^3$
46. $g(t) = e^{-3/t^2}$
47. $y = \ln(e^{x^2})$
48. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$
49. $y = \ln(1 + e^{2x})$
50. $y = \ln \dfrac{e^x + e^{-x}}{2}$
51. $y = \dfrac{2}{e^x + e^{-x}}$
52. $y = \dfrac{e^x - e^{-x}}{2}$
53. $y = x^2e^x - 2xe^x + 2e^x$
54. $y = xe^x - e^x$
55. $f(x) = e^{-x}\ln x$
56. $f(x) = e^3 \ln x$
57. $y = e^x(\sin x + \cos x)$
58. $y = \ln e^x$

In Exercises 59 and 60, use implicit differentiation to find dy/dx.

59. $xe^y - 10x + 3y = 0$
60. $e^{xy} + x^2 - y^2 = 10$

In Exercises 61 and 62, find the second derivative of the function.

61. $f(x) = (3 + 2x)e^{-3x}$
62. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 63 and 64, show that the function $y = f(x)$ is a solution of the differential equation.

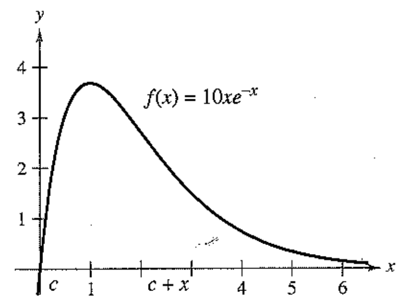
63. $y = e^x(\cos\sqrt{2}x + \sin\sqrt{2}x)$
$y'' - 2y' + 3y = 0$
64. $y = e^x(3\cos 2x - 4\sin 2x)$
$y'' - 2y' + 5y = 0$

In Exercises 65–72, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$
66. $f(x) = \dfrac{e^x - e^{-x}}{2}$
67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$
68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$
69. $f(x) = x^2e^{-x}$
70. $f(x) = xe^{-x}$
71. $g(t) = 1 + (2 + t)e^{-t}$
72. $f(x) = -2 + e^{3x}(4 - 2x)$

73. ***Area*** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

74. ***Area*** Perform the following steps to find the maximum area of the rectangle shown in the figure.
(a) Solve for c in the equation $f(c) = f(c + x)$.
(b) Use the result in part (a) to write the area A as a function of x. [*Hint:* $A = xf(c)$]
(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.
(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate
$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$
Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.



75. Verify that the function
$$y = \frac{L}{1 + ae^{-x/b}}, \qquad a > 0, b > 0, L > 0$$
increases at a maximum rate when $y = L/2$.

76. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin. (Use Newton's Method or the root-finding capabilities of a graphing utility.)

77. Find, to three decimal places, the value of x such that
$e^{-x} = x$.
(Use Newton's Method or the root-finding capabilities of a graphing utility.)

78. ***Depreciation*** The value V of an item t years after it is purchased is
$V = 15{,}000e^{-0.6286t}, \qquad 0 \le t \le 10.$
(a) Use a graphing utility to graph the function.
(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 5$.
(c) Use a graphing utility to graph the tangent line to the function when $t = 1$ and $t = 5$.

79. ***Writing*** Consider the function
$$f(x) = \frac{2}{1 + e^{1/x}}.$$
(a) Use a graphing utility to graph f.
(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

80. ***Harmonic Motion*** The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is

$$y = 1.56e^{-0.22t}\cos 4.9t$$

where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

81. ***Modeling Data*** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

82. ***Modeling Data*** A 1994 Chevrolet Camaro coupe with a 6-cylinder engine, 5-speed transmission, and air conditioning had a retail price of \$17,040. A local dealership had the following guide for the approximate value of the car for the years 1995 through 2000.

Year	1994	1995	1996	1997
Value	\$17,040	\$14,590	\$12,845	\$10,995

Year	1998	1999	2000
Value	\$9,220	\$8,095	\$6,835

In each of the following, let V represent the value of the automobile in the year t, with $t = 4$ corresponding to 1994.

(a) Use a computer algebra system to find linear and quadratic models for the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use a computer algebra system to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Find the rate of decrease in the value of the car when $t = 5$ and $t = 9$ using the exponential model.

Linear and Quadratic Approximations **In Exercises 83 and 84, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

83. $f(x) = e^{x/2}$

84. $f(x) = e^{-x^2/2}$

85. ***Finding a Pattern*** Use a graphing utility to compare the graph of the function $y = e^x$ with the graphs of each of the following functions.

(a) $y_1 = 1 + \dfrac{x}{1!}$

(b) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(c) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

86. Identify the pattern of successive polynomials in Exercise 85. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

In Exercises 87–108, find or evaluate the integral.

87. $\displaystyle\int e^{5x}(5)\,dx$

88. $\displaystyle\int e^{-x^4}(-4x^3)\,dx$

89. $\displaystyle\int_0^1 e^{-2x}\,dx$

90. $\displaystyle\int_3^4 e^{3-x}\,dx$

91. $\displaystyle\int xe^{-x^2}\,dx$

92. $\displaystyle\int x^2e^{x^3/2}\,dx$

93. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

94. $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx$

95. $\displaystyle\int \frac{e^{-x}}{1 + e^{-x}}\,dx$

96. $\displaystyle\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

97. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

98. $\displaystyle\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

99. $\displaystyle\int e^x\sqrt{1 - e^x}\,dx$

100. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

101. $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

102. $\displaystyle\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

103. $\displaystyle\int \frac{5 - e^x}{e^{2x}}\,dx$

104. $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

105. $\displaystyle\int e^{\sin \pi x}\cos \pi x\,dx$

106. $\displaystyle\int e^{\sec 2x}\sec 2x\tan 2x\,dx$

107. $\displaystyle\int e^{-x}\tan(e^{-x})\,dx$

108. $\displaystyle\int \ln(e^{2x-1})\,dx$

In Exercises 109 and 110, solve the differential equation.

109. $\dfrac{dy}{dx} = xe^{ax^2}$

110. $\dfrac{dy}{dx} = (e^x - e^{-x})^2$

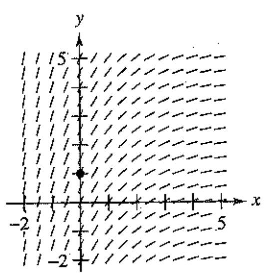
In Exercises 111 and 112, find the particular solution that satisfies the initial conditions.

111. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

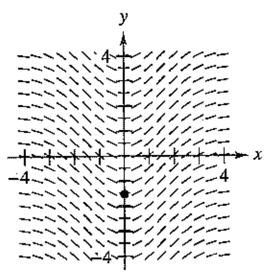
112. $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Slope Fields **In Exercises 113 and 114, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

113. $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$



114. $\frac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\frac{3}{2}\right)$



Area **In Exercises 115–118, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

115. $y = e^x, y = 0, x = 0, x = 5$

116. $y = e^{-x}, y = 0, x = a, x = b$

117. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

118. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

119. Given the exponential function $f(x) = e^x$, show that

(a) $f(u - v) = \frac{f(u)}{f(v)}$. (b) $f(kx) = [f(x)]^k$.

120. Approximate each integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Then use the integration capabilities of a graphing utility to approximate the integrals and compare the results.

(a) $\int_0^4 \sqrt{x}\, e^x\, dx$ (b) $\int_0^2 2xe^{-x}\, dx$

121. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2}\, dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

122. ***Probability*** The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation

$$\int_0^x 0.3e^{-0.3t}\, dt = \frac{1}{2}.$$

Solve the equation.

123. Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t\, dt \geq \int_0^x 1\, dt.$$

Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

124. ***Modeling Data*** A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use the definite integral to approximate the number of liters of chemical released during the 4 hours.

Getting at the Concept

125. In your own words, state the properties of the natural exponential function.

126. Describe the relationship between the graph of $f(x) = \ln x$ and $g(x) = e^x$.

127. Is there a function f such that $f(x) = f'(x)$? If so, identify it.

128. Without integrating, state the integration formula you can use to integrate each of the following.

(a) $\int \frac{e^x}{e^x + 1}\, dx$ (b) $\int xe^{x^2}\, dx$

129. Explain why $\int_0^2 e^{-x}\, dx > 0$.

130. Prove that $\frac{e^a}{e^b} = e^{a-b}$.

131. Let $f(x) = \frac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

EXERCISES FOR SECTION 5.5

Depreciation **In Exercises 1–4, the time in which a machine depreciates to one-half its purchase price is given. Find a model that yields the fraction of the purchase price as a function of time and determine that fraction at time t_0.**

	Depreciation time	t_0
1.	3 years	6 years
2.	8 years	16 years
3.	7 years	10 years
4.	5 years	2 years

In Exercises 5–8, evaluate the expression without using a calculator.

5. $\log_2 \frac{1}{8}$ **6.** $\log_{27} 9$

7. $\log_7 1$ **8.** $\log_a \dfrac{1}{a}$

In Exercises 9–12, write the exponential equation as a logarithmic equation or vice versa.

9. (a) $2^3 = 8$ (b) $3^{-1} = \frac{1}{3}$

10. (a) $27^{2/3} = 9$ (b) $16^{3/4} = 8$

11. (a) $\log_{10} 0.01 = -2$ (b) $\log_{0.5} 8 = -3$

12. (a) $\log_3 \frac{1}{9} = -2$ (b) $49^{1/2} = 7$

In Exercises 13–18, sketch the graph of the function by hand.

13. $y = 3^x$ **14.** $y = 3^{x-1}$

15. $y = \left(\frac{1}{3}\right)^x$ **16.** $y = 2^{x^2}$

17. $h(x) = 5^{x-2}$ **18.** $y = 3^{-|x|}$

In Exercises 19–24, solve for x or b.

19. (a) $\log_{10} 1000 = x$ (b) $\log_{10} 0.1 = x$

20. (a) $\log_3 \frac{1}{81} = x$ (b) $\log_6 36 = x$

21. (a) $\log_3 x = -1$ (b) $\log_2 x = -4$

22. (a) $\log_b 27 = 3$ (b) $\log_b 125 = 3$

23. (a) $x^2 - x = \log_5 25$ (b) $3x + 5 = \log_2 64$

24. (a) $\log_3 x + \log_3(x - 2) = 1$ (b) $\log_{10}(x + 3) - \log_{10} x = 1$

In Exercises 25–34, solve the equation accurate to three decimal places.

25. $3^{2x} = 75$ **26.** $5^{6x} = 8320$

27. $2^{3-z} = 625$ **28.** $3(5^{x-1}) = 86$

29. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$ **30.** $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$

31. $\log_2(x - 1) = 5$ **32.** $\log_{10}(t - 3) = 2.6$

33. $\log_3 x^2 = 4.5$ **34.** $\log_5\sqrt{x - 4} = 3.2$

In Exercises 35–38, use a graphing utility to graph the function and approximate its zero(s) accurate to three decimal places.

35. $g(x) = 6(2^{1-x}) - 25$

36. $f(t) = 300(1.0075^{12t}) - 735.41$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

In Exercises 39 and 40, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

39. $f(x) = 4^x$, $g(x) = \log_4 x$

40. $f(x) = 3^x$, $g(x) = \log_3 x$

In Exercises 41–56, find the derivative of the function.

41. $f(x) = 4^x$ **42.** $g(x) = 2^{-x}$

43. $y = 5^{x-2}$ **44.** $y = x(6^{-2x})$

45. $g(t) = t^2 2^t$ **46.** $f(t) = \dfrac{3^{2t}}{t}$

47. $h(\theta) = 2^{-\theta} \cos \pi\theta$ **48.** $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

49. $y = \log_3 x$ **50.** $y = \log_{10} 2x$

51. $f(x) = \log_2 \dfrac{x^2}{x - 1}$ **52.** $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

53. $y = \log_5 \sqrt{x^2 - 1}$ **54.** $y = \log_{10} \dfrac{x^2 - 1}{x}$

55. $g(t) = \dfrac{10 \log_4 t}{t}$ **56.** $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

In Exercises 57–60, use logarithmic differentiation to find dy/dx.

57. $y = x^{2/x}$

58. $y = x^{x-1}$

59. $y = (x - 2)^{x+1}$

60. $y = (1 + x)^{1/x}$

In Exercises 61–68, find or evaluate the integral.

61. $\int 3^x \, dx$ **62.** $\int 5^{-x} \, dx$

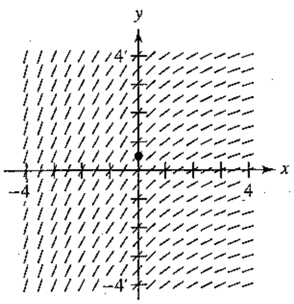
63. $\int_{-1}^{2} 2^x \, dx$ **64.** $\int_{-2}^{0} (3^3 - 5^2) \, dx$

65. $\int x(5^{-x^2}) \, dx$ **66.** $\int (3 - x)7^{(3-x)^2} \, dx$

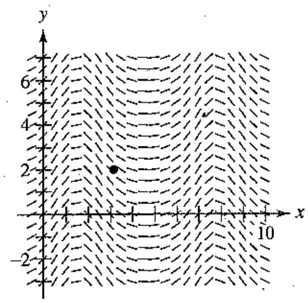
67. $\int \dfrac{3^{2x}}{1 + 3^{2x}} \, dx$ **68.** $\int 2^{\sin x} \cos x \, dx$

Slope Fields **In Exercises 69 and 70, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

69. $\frac{dy}{dx} = 0.4^{x/3}$, $\left(0, \frac{1}{2}\right)$



70. $\frac{dy}{dx} = e^{\sin x} \cos x$, $(\pi, 2)$



Getting at the Concept

71. List some applications of the exponential functions $f(x) = a^x$ and $g(x) = a^{-x}$.

72. Describe how to use a calculator to find the logarithm of a number if the base is not 10 or e.

73. The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

x	1	2	8
y	0	1	3

74. Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) If x is a real number between 1000 and 10,000, determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) If $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

75. ***Ordering Functions*** Order the functions

$$f(x) = \log_2 x,\ g(x) = x^x,\ h(x) = x^2, \text{ and } k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the smallest rate of growth for "large" values of x.

76. Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

77. ***Inflation*** If the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) If the price of an oil change for your car is presently $24.95, estimate the price 10 years from now.

(b) Find the rate of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

78. ***Depreciation*** After t years, the value of a car purchased for $20,000 is

$$V(t) = 20{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 79–82, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous compounding
A						

79. $P = \$1000$
$r = 3\frac{1}{2}\%$
$t = 10$ years

80. $P = \$2500$
$r = 6\%$
$t = 20$ years

81. $P = \$1000$
$r = 5\%$
$t = 30$ years

82. $P = \$5000$
$r = 7\%$
$t = 25$ years

Compound Interest **In Exercises 83–86, complete the table to determine the amount of money P (present value) that should be invested at rate r to produce a balance of \$100,000 in t years.**

t	1	10	20	30	40	50
P						

83. $r = 5\%$
Compounded continuously

84. $r = 6\%$
Compounded continuously

85. $r = 5\%$
Compounded monthly

86. $r = 7\%$
Compounded daily

87. ***Compound Interest*** Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now

(b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

88. ***Compound Interest*** Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a graphing utility to graph the exponential functions giving the growth of the investment over the 20 years for each of the following interest rates. Compare the ending balances for each of the rates.

(a) $r = 3\%$ (b) $r = 5\%$ (c) $r = 6\%$

89. ***Timber Yield*** The yield V (in millions of cubic feet per acre) for a stand of timber at age t is

$$V = 6.7e^{(-48.1)/t}$$

where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rate at which the yield is changing when $t = 20$ years and $t = 60$ years.

90. ***Learning Theory*** In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.86}{1 + e^{-0.25n}}.$$

(a) Find the limiting proportion of correct responses as n approaches infinity.

(b) Find the rate at which P is changing after $n = 3$ trials and $n = 10$ trials.

91. ***Forest Defoliation*** To estimate the amount of defoliation caused by the gypsy moth during a year, a forester counts the number of egg masses on $\frac{1}{40}$ of an acre the preceding fall. The percent of defoliation y is approximated by

$$y = \frac{300}{3 + 17e^{-0.0625x}}$$

where x is the number of egg masses in thousands. *(Source: USDA Forest Service)*

(a) Use a graphing utility to graph the function.

(b) Estimate the percent of defoliation if 2000 egg masses are counted.

(c) Estimate the number of egg masses that existed if you observe that approximately $\frac{2}{3}$ of a forest is defoliated.

(d) Use calculus to estimate the value of x for which y is increasing most rapidly.

92. ***Population Growth*** A lake is stocked with 500 fish, and their population increases according to the logistics curve

$$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

93. ***Modeling Data*** The breaking strength B (in tons) of a steel cable of diameter d (in inches) is given in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rate of growth of the model when $d = 0.8$ and $d = 1.5$.

94. ***Comparing Models*** The amount y (in billions of dollars) given to philanthropy (from individuals, foundations, corporations, and charitable bequests) in the United States for the years 1991 through 1997 is given in the table, with $x = 1$ corresponding to 1991. *(Source: AAFRC Trust for Philanthropy)*

x	1	2	3	4	5	6	7
y	105.0	110.4	116.5	119.2	124.3	133.5	143.5

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$ $\quad y_2 = a + b \ln x$

$y_3 = ab^x$ $\quad y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 1996. Which model is increasing at the greatest rate in 1996?

95. ***Conjecture***

(a) Use a graphing utility to approximate the integrals of the functions

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3}, \quad g(t) = 4\left(\frac{\sqrt[3]{9}}{4}\right)^{t}, \text{ and } h(t) = 4e^{-0.653886t}$$

on the interval $[0, 4]$.

(b) Use a graphing utility to graph the three functions.

(c) Use the results in parts (a) and (b) to make a conjecture about the three functions. Could you make the conjecture using only part (a)? Explain. Prove your conjecture analytically.

96. ***Area*** Find the area of the region bounded by the graphs of $y = 3^x$, $y = 0$, $x = 0$, and $x = 3$.

97. ***Continuous Cash Flow*** The present value P of a continuous cash flow of \$2000 per year earning 6% interest compounded continuously over 10 years is

$$P = \int_0^{10} 2000e^{-0.06t}\, dt.$$

Find P.

98. Complete the table to demonstrate that e can also be defined as $\lim_{x\to 0^+} (1 + x)^{1/x}$.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

In Exercises 99 and 100, find an exponential function that fits the experimental data collected over time *t*.

99.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

100.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

True or False? **In Exercises 101–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. $e = 271{,}801/99{,}990$.

102. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

103. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inverse functions of each other.

104. The exponential function $y = Ce^x$ is a solution of the differential equation $d^n y/dx^n = y$, $n = 1, 2, 3, \ldots$.

105. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

106. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

107. Solve the logistics differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

and obtain the logistics growth function of Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

108. Find an equation of the tangent line to $y = x^{\sin x}$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

SECTION PROJECT USING GRAPHING UTILITIES TO ESTIMATE SLOPE

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \le x \le 3$, $-2 \le y \le 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x\to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer with a graphing utility.

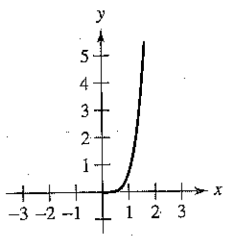
FOR FURTHER INFORMATION For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

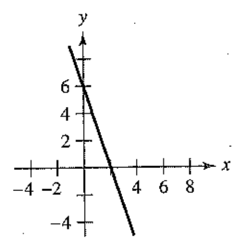
EXERCISES FOR SECTION 5.3

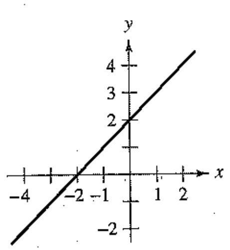
In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.

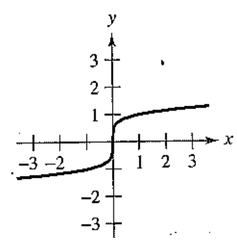
1. $f(x) = 5x + 1$, $\quad g(x) = (x - 1)/5$
2. $f(x) = 3 - 4x$, $\quad g(x) = (3 - x)/4$
3. $f(x) = x^3$, $\quad g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3$, $\quad g(x) = \sqrt[3]{1 - x}$
5. $f(x) = \sqrt{x - 4}$, $\quad g(x) = x^2 + 4, \quad x \geq 0$
6. $f(x) = 16 - x^2, \quad x \geq 0$, $\quad g(x) = \sqrt{16 - x}$
7. $f(x) = 1/x$, $\quad g(x) = 1/x$
8. $f(x) = \dfrac{1}{1 + x}, \quad x \geq 0$, $\quad g(x) = \dfrac{1 - x}{x}, \quad 0 < x \leq 1$

In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


(d)


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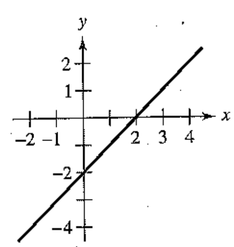

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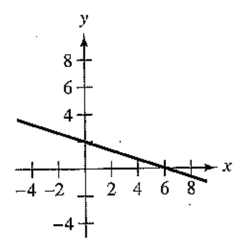

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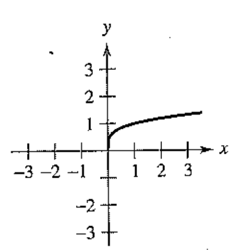

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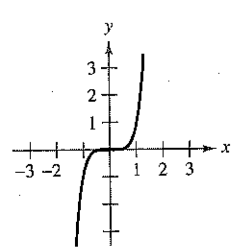
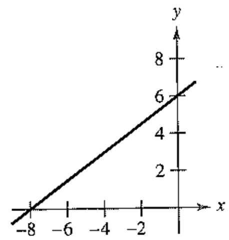
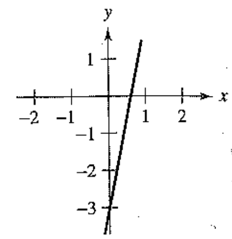


In Exercises 13–16, use the horizontal line test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

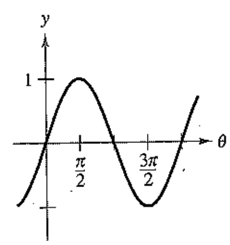
13. $f(x) = \frac{3}{4}x + 6$



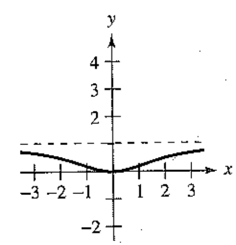
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin \theta$



16. $f(x) = \dfrac{x^2}{x^2 + 4}$



In Exercises 17–22, use a graphing utility to graph the function. Determine whether the function is one-to-one on its entire domain.

17. $h(s) = \dfrac{1}{s - 2} - 3$
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$
19. $f(x) = \ln x$
20. $f(x) = 5x\sqrt{x - 1}$
21. $g(x) = (x + 5)^3$
22. $h(x) = |x + 4| - |x - 4|$

In Exercises 23–28, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.

23. $f(x) = (x + a)^3 + b$
24. $f(x) = \cos \dfrac{3x}{2}$
25. $f(x) = \dfrac{x^4}{4} - 2x^2$
26. $f(x) = x^3 - 6x^2 + 12x$
27. $f(x) = 2 - x - x^3$
28. $f(x) = \ln(x - 3)$

In Exercises 29–36, find the inverse function of f. Graph (by hand) f and f^{-1}. Describe the relationship between the graphs.

29. $f(x) = 2x - 3$
30. $f(x) = 3x$
31. $f(x) = x^5$
32. $f(x) = x^3 - 1$
33. $f(x) = \sqrt{x}$
34. $f(x) = x^2, \quad x \geq 0$
35. $f(x) = \sqrt{4 - x^2}, \quad x \geq 0$
36. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$

In Exercises 37–42, find the inverse function of f. Use a graphing utility to graph f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

37. $f(x) = \sqrt[3]{x-1}$

38. $f(x) = 3\sqrt[5]{2x-1}$

39. $f(x) = x^{2/3}, \quad x \geq 0$

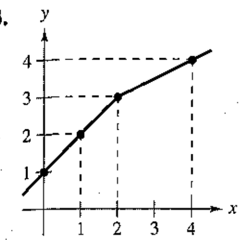
40. $f(x) = x^{3/5}$

41. $f(x) = \dfrac{x}{\sqrt{x^2+7}}$

42. $f(x) = \dfrac{x+2}{x}$

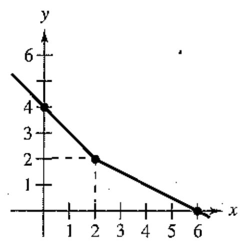
In Exercises 43 and 44, use the graph of the function f to complete the table and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

43.



x	1	2	3	4
$f^{-1}(x)$				

44.



x	0	2	4
$f^{-1}(x)$			

45. ***Cost*** Suppose you need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is

$$y = 1.25x + 1.60(50 - x)$$

where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) Use the context of the problem to determine the domain of the inverse function.

(d) Determine the number of pounds of the less expensive commodity purchased if the total cost is \$73.

46. ***Think About It*** The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find k.

In Exercises 47–52, show that f is strictly monotonic on the indicated interval and therefore has an inverse function on that interval.

Function	*Interval*
47. $f(x) = (x-4)^2$	$[4, \infty)$
48. $f(x) = \lvert x+2 \rvert$	$[-2, \infty)$
49. $f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
50. $f(x) = \cot x$	$(0, \pi)$
51. $f(x) = \cos x$	$[0, \pi]$
52. $f(x) = \sec x$	$\left[0, \dfrac{\pi}{2}\right)$

In Exercises 53 and 54, find the inverse function of f over the indicated interval. Use a graphing utility to graph f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

Function	*Interval*
53. $f(x) = \dfrac{x}{x^2-4}$	$-2 < x < 2$
54. $f(x) = 2 - \dfrac{3}{x^2}$	$0 < x < 10$

Graphical Reasoning **In Exercises 55–58, (a) use a graphing utility to graph the function, (b) use the *drawing* feature of a graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function. Explain your reasoning.**

55. $f(x) = x^3 + x + 4$

56. $h(x) = x\sqrt{4 - x^2}$

57. $g(x) = \dfrac{3x^2}{x^2+1}$

58. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$

In Exercises 59–62, determine whether the function is one-to-one. If it is, find its inverse function.

59. $f(x) = \sqrt{x-2}$

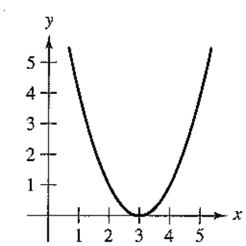
60. $f(x) = -3$

61. $f(x) = \lvert x - 2 \rvert, \quad x \leq 2$

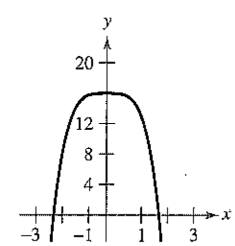
62. $f(x) = ax + b, \quad a \neq 0$

In Exercises 63–66, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)

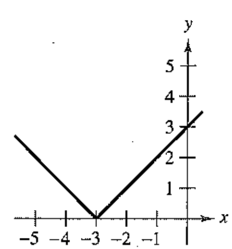
63. $f(x) = (x-3)^2$



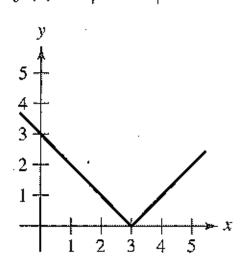
64. $f(x) = 16 - x^4$



65. $f(x) = \lvert x + 3 \rvert$



66. $f(x) = \lvert x - 3 \rvert$



Think About It **In Exercises 67–70, decide whether the function has an inverse function. If so, what is the inverse function?**

67. $g(t)$ is the volume of water that has passed through a water line t minutes after a control valve is opened.

68. $h(t)$ is the height of the tide t hours after midnight, where $0 \le t < 24$.

69. $C(t)$ is the cost of a long distance call lasting t minutes.

70. $A(r)$ is the area of a circle of radius r.

In Exercises 71–76, find $(f^{-1})'(a)$ for the function f and real number a.

Function	*Real Number*
71. $f(x) = x^3 + 2x - 1$	$a = 2$
72. $f(x) = \frac{1}{27}(x^5 + 2x^3)$,	$a = -11$
73. $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$a = \frac{1}{2}$
74. $f(x) = \cos 2x, \quad 0 \le x \le \frac{\pi}{2}$	$a = 1$
75. $f(x) = x^3 - \frac{4}{x}$	$a = 6$
76. $f(x) = \sqrt{x - 4}$	$a = 2$

In Exercises 77–80, (a) find the domains of f and f^{-1}, (b) find the ranges of f and f^{-1}, (c) graph f and f^{-1}, and (d) show that the slopes of the graphs of f and f^{-1} are reciprocals at the indicated points.

Functions	*Point*
77. $f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
78. $f(x) = 3 - 4x$	$(1, -1)$
$f^{-1}(x) = \frac{3 - x}{4}$	$(-1, 1)$
79. $f(x) = \sqrt{x - 4}$	$(5, 1)$
$f^{-1}(x) = x^2 + 4, \quad x \ge 0$	$(1, 5)$
80. $f(x) = \frac{4}{1 + x^2}, \quad x \ge 0$	$(1, 2)$
$f^{-1}(x) = \sqrt{\frac{4 - x}{x}}$	$(2, 1)$

In Exercises 81 and 82, find dy/dx at the indicated point for the equation.

81. $x = y^3 - 7y^2 + 2$
$(-4, 1)$

82. $x = 2 \ln(y^2 - 3)$
$(0, 4)$

In Exercises 83–86, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value.

83. $(f^{-1} \circ g^{-1})(1)$

84. $(g^{-1} \circ f^{-1})(-3)$

85. $(f^{-1} \circ f^{-1})(6)$

86. $(g^{-1} \circ g^{-1})(-4)$

In Exercises 87–90, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the indicated function.

87. $g^{-1} \circ f^{-1}$

88. $f^{-1} \circ g^{-1}$

89. $(f \circ g)^{-1}$

90. $(g \circ f)^{-1}$

Getting at the Concept

91. Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example.

92. Describe the relationship between the graph of a function and the graph of its inverse function.

93. Give an example of a function that does *not* have an inverse function.

94. State the theorem that gives the method for finding the derivative of an inverse function.

In Exercises 95 and 96, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain.

95. $f(x) = \tan x$

96. $f(x) = \frac{x}{x^2 - 4}$

97. Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

98. Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

99. Prove that if a function has an inverse function, then the inverse function is unique.

100. Prove that a function has an inverse function if and only if it is one-to-one.

True or False? **In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. If f is an even function, then f^{-1} exists.

102. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

103. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

104. There exists no function f such that $f = f^{-1}$.

105. Is the converse of the second part of Theorem 5.7 true? That is, if a function is one-to-one (and hence has an inverse function), then must the function be strictly monotonic? If so, prove it. If not, give a counterexample.

106. Let f be twice-differentiable and one-to-one on an open interval I. Show that its inverse function g satisfies

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

If f is increasing and concave downward, what is the concavity of $f^{-1} = g$?

107. If $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}$, find $(f^{-1})'(0)$.

EXERCISES FOR SECTION 5.8

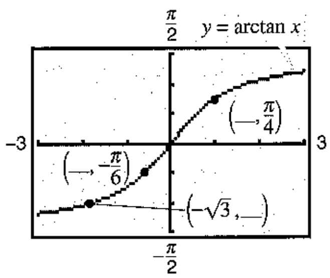
Numerical and Graphical Analysis **In Exercises 1 and 2, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.**

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

1. $y = \arcsin x$
2. $y = \arccos x$

3. ***True or False?*** Decide whether the following statement is true or false, and explain: Because $\cos(-\pi/3) = \frac{1}{2}$, it follows that $\arccos \frac{1}{2} = -\pi/3$.

4. Determine the missing coordinates of the points on the graph of the function.



In Exercises 5–12, evaluate the expression without using a calculator.

5. $\arcsin \frac{1}{2}$
6. $\arcsin 0$
7. $\arccos \frac{1}{2}$
8. $\arccos 0$
9. $\arctan \frac{\sqrt{3}}{3}$
10. $\operatorname{arccot}(-\sqrt{3})$
11. $\operatorname{arccsc}(-\sqrt{2})$
12. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

In Exercises 13–16, use a calculator to approximate the value. Round your answer to two decimal places.

13. $\arccos(-0.8)$
14. $\arcsin(-0.39)$
15. $\operatorname{arcsec} 1.269$
16. $\arctan(-3)$

In Exercises 17–20, evaluate the expression without using a calculator. (*Hint:* See Example 3.)

17. (a) $\sin\left(\arctan \frac{3}{4}\right)$
(b) $\sec\left(\arcsin \frac{4}{5}\right)$

18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$
(b) $\cos\left(\arcsin \frac{5}{13}\right)$

19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

In Exercises 21–28, write the expression in algebraic form.

21. $\cos(\arcsin 2x)$
22. $\sec(\arctan 4x)$
23. $\sin(\operatorname{arcsec} x)$
24. $\cos(\operatorname{arccot} x)$
25. $\tan\left(\operatorname{arcsec} \frac{x}{3}\right)$
26. $\sec[\arcsin(x - 1)]$
27. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$
28. $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 29 and 30, use a graphing utility to graph f and g in the same viewing window to verify that they are equal. Explain why they are equal. Identify any asymptotes of the graphs.

29. $f(x) = \sin(\arctan 2x), \quad g(x) = \frac{2x}{\sqrt{1 + 4x^2}}$

30. $f(x) = \tan\left(\arccos \frac{x}{2}\right), \quad g(x) = \frac{\sqrt{4 - x^2}}{x}$

In Exercises 31–34, solve the equation for x.

31. $\arcsin(3x - \pi) = \frac{1}{2}$
32. $\arctan(2x - 5) = -1$
33. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$
34. $\arccos x = \operatorname{arcsec} x$

In Exercises 35 and 36, verify each identity.

35. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, \quad x \geq 1$
(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, \quad x > 0$

36. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$
(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

In Exercises 37–40, sketch the graph of the function. Use a graphing utility to verify your graph.

37. $f(x) = \arcsin(x - 1)$
38. $f(x) = \arctan x + \frac{\pi}{2}$
39. $f(x) = \operatorname{arcsec} 2x$
40. $f(x) = \arccos \frac{x}{4}$

In Exercises 41–60, find the derivative of the function.

41. $f(x) = 2\arcsin(x - 1)$
42. $f(t) = \arcsin t^2$
43. $g(x) = 3\arccos \frac{x}{2}$
44. $f(x) = \operatorname{arcsec} 2x$
45. $f(x) = \arctan \frac{x}{a}$
46. $f(x) = \arctan\sqrt{x}$
47. $g(x) = \frac{\arcsin 3x}{x}$
48. $h(x) = x^2 \arctan x$
49. $h(t) = \sin(\arccos t)$
50. $f(x) = \arcsin x + \arccos x$
51. $y = x \arccos x - \sqrt{1 - x^2}$
52. $y = \ln(t^2 + 4) - \frac{1}{2}\arctan \frac{t}{2}$
53. $y = \frac{1}{2}\left(\frac{1}{2}\ln \frac{x + 1}{x - 1} + \arctan x\right)$

54. $y = \frac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right)\right]$

55. $y = x \arcsin x + \sqrt{1 - x^2}$

56. $y = x \arctan 2x - \frac{1}{4}\ln(1 + 4x^2)$

57. $y = 8 \arcsin\frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2}$

58. $y = 25 \arcsin\frac{x}{5} - x\sqrt{25 - x^2}$

59. $y = \arctan x + \frac{x}{1 + x^2}$

60. $y = \arctan\frac{x}{2} - \frac{1}{2(x^2 + 4)}$

Linear and Quadratic Approximations **In Exercises 61 and 62, use a computer algebra system to find the linear approximation**

$P_1(x) = f(a) + f'(a)(x - a)$

and the quadratic approximation

$P_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$

to the function f at $x = a$. Sketch the graph of the function and its linear and quadratic approximations.

61. $f(x) = \arcsin x$, $a = \frac{1}{2}$

62. $f(x) = \arctan x$, $a = 1$

In Exercises 63–66, find any relative extrema of the function.

63. $f(x) = \operatorname{arcsec} x - x$

64. $f(x) = \arcsin x - 2x$

65. $f(x) = \arctan x - \arctan(x - 4)$

66. $h(x) = \arcsin x - 2 \arctan x$

Getting at the Concept

67. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

68. Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

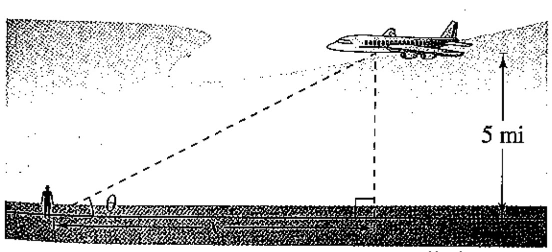
69. Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

70. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

71. ***Angular Rate of Change*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) If the speed of the plane is 400 miles per hour, find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.



Not drawn to scale

72. ***Writing*** Repeat Exercise 71 if the altitude of the plane is 3 miles and describe how the altitude affects the rate of change of θ.

73. ***Angular Rate of Change*** In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object.

(a) Find the position function giving the height of the object at time t assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rate of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

74. ***Angular Rate of Change*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let s be the distance between the camera and the shuttle. Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

75. Prove that

$$\arctan x + \arctan y = \arctan\frac{x + y}{1 - xy}, \quad xy \neq 1.$$

Use this formula to show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

76. Verify each differentiation formula.

(a) $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$ (b) $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$

(c) $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$

(d) $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$

(e) $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1 + u^2}$ (f) $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

77. ***Existence of an Inverse*** Determine the values of k such that the function $f(x) = kx + \sin x$ has an inverse function.

78. ***Think About It*** Use a graphing utility to graph

$f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Why isn't the graph of g the line $y = x$?

(b) Determine the extrema of g.

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

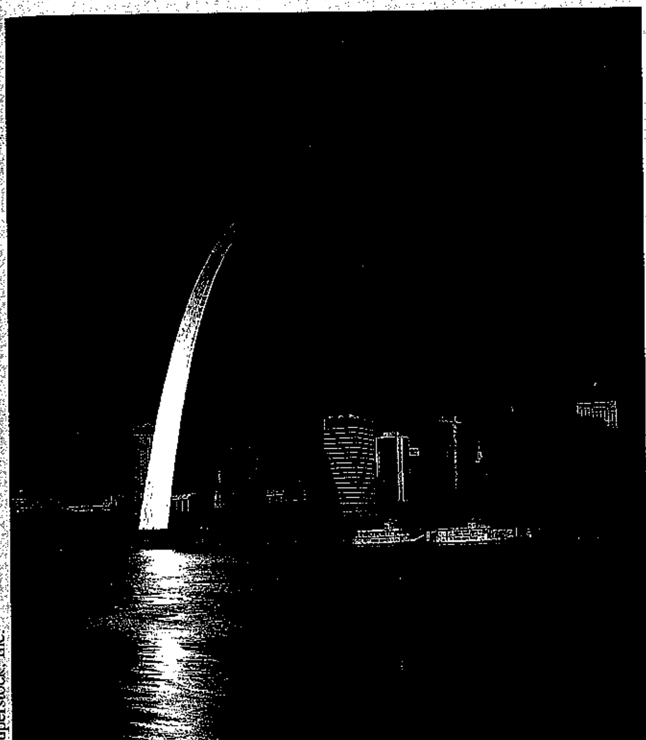
79. The slope of the graph of the inverse tangent function is positive for all x.

80. The range of $y = \arcsin x$ is $[0, \pi]$.

81. $\frac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

82. $\arcsin^2 x + \arccos^2 x = 1$

SECTION PROJECT ST. LOUIS ARCH



Superstock, Inc.

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used to construct the arch was

$$y = 693.8597 - 68.7672 \cosh 0.0100333x,$$
$$-299.2239 \le x \le 299.2239$$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is

$$A = 125.1406 \cosh 0.0100333x.$$

(Source: Owner's Manual for the Gateway Arch, *Saint Louis, MO, by William Thayer)*

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?

REVIEW EXERCISES FOR CHAPTER 5

5.1 In Exercises 1 and 2, sketch the graph of the function by hand. Identify any asymptotes of the graph.

1. $f(x) = \ln x + 3$

2. $f(x) = \ln(x - 3)$

In Exercises 3 and 4, use the properties of logarithms to expand the logarithmic function.

3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$

4. $\ln[(x^2 + 1)(x - 1)]$

In Exercises 5 and 6, write the expression as the logarithm of a single quantity.

5. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$

6. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

In Exercises 7 and 8, solve the equation for x.

7. $\ln \sqrt{x + 1} = 2$

8. $\ln x + \ln(x - 3) = 0$

In Exercises 9–16, find the derivative of the function.

9. $g(x) = \ln \sqrt{x}$

10. $h(x) = \ln \dfrac{x(x - 1)}{x - 2}$

11. $f(x) = x\sqrt{\ln x}$

12. $f(x) = \ln[x(x^2 - 2)^{2/3}]$

13. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

14. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

15. $y = -\dfrac{1}{a}\ln \dfrac{a + bx}{x}$

16. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln \dfrac{a + bx}{x}$

5.2 In Exercises 17–24, find or evaluate the integral.

17. $\displaystyle\int \frac{1}{7x - 2}\,dx$

18. $\displaystyle\int \frac{x}{x^2 - 1}\,dx$

19. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx$

20. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\,dx$

21. $\displaystyle\int_1^4 \frac{x + 1}{x}\,dx$

22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

23. $\displaystyle\int_0^{\pi/3} \sec \theta\, d\theta$

24. $\displaystyle\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx$

5.3 **In Exercises 25–30, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.**

25. $f(x) = \frac{1}{2}x - 3$ **26.** $f(x) = 5x - 7$

27. $f(x) = \sqrt{x+1}$ **28.** $f(x) = x^3 + 2$

29. $f(x) = \sqrt[3]{x+1}$ **30.** $f(x) = x^2 - 5, \quad x \geq 0$

In Exercise 31–34, find $(f^{-1})'(a)$ for the function f and real number a.

	Function	*Real number*
31.	$f(x) = x^3 + 2$	$a = -1$
32.	$f(x) = x\sqrt{x-3}$	$a = 4$
33.	$f(x) = \tan x, \ -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$	$a = \frac{\sqrt{3}}{3}$
34.	$f(x) = \ln x$	$a = 0$

5.4 **In Exercises 35 and 36, (a) find the inverse function of f, (b) using a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.**

35. $f(x) = \ln\sqrt{x}$ **36.** $f(x) = e^{1-x}$

In Exercises 37 and 38, graph the function without the aid of a graphing utility.

37. $y = e^{-x/2}$ **38.** $y = 4e^{-x^2}$

In Exercises 39–46, find the derivative of the function.

39. $f(x) = \ln(e^{-x^2})$ **40.** $g(x) = \ln\frac{e^x}{1+e^x}$

41. $g(t) = t^2e^t$ **42.** $h(z) = e^{-z^2/2}$

43. $y = \sqrt{e^{2x} + e^{-2x}}$ **44.** $y = 3e^{-3/t}$

45. $g(x) = \frac{x^2}{e^x}$ **46.** $f(\theta) = \frac{1}{2}e^{\sin 2\theta}$

In Exercises 47 and 48, use implicit differentiation to find dy/dx.

47. $y\ln x + y^2 = 0$ **48.** $\cos x^2 = xe^y$

In Exercises 49–56, find the indefinite integral.

49. $\int xe^{-3x^2}\,dx$ **50.** $\int \frac{e^{1/x}}{x^2}\,dx$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$ **52.** $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

53. $\int xe^{1-x^2}\,dx$ **54.** $\int x^2e^{x^3+1}\,dx$

55. $\int \frac{e^x}{e^x - 1}\,dx$ **56.** $\int \frac{e^{2x}}{e^{2x} + 1}\,dx$

57. Show that $y = e^x(a\cos 3x + b\sin 3x)$ satisfies the differential equation $y'' - 2y' + 10y = 0$.

58. ***Depreciation*** The value V of an item t years after it is purchased is

$$V = 8000e^{-0.6t}, \quad 0 \leq t \leq 5.$$

(a) Use a graphing utility to graph the function.

(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to sketch the tangent line to the function when $t = 1$ and $t = 4$.

In Exercises 59 and 60, find the area of the region bounded by the graphs of the equations.

59. $y = xe^{-x^2}, \ y = 0, \ x = 0, \ x = 4$

60. $y = 2e^{-x}, \ y = 0, \ x = 0, \ x = 2$

5.5 **In Exercises 61–64, sketch the graph of the function by hand.**

61. $y = 3^{x/2}$ **62.** $y = 6(2^{-x^2})$

63. $y = \log_2(x - 1)$ **64.** $y = \log_4 x^2$

In Exercises 65–70, find the derivative of the function.

65. $f(x) = 3^{x-1}$ **66.** $f(x) = (4e)^x$

67. $y = x^{2x+1}$ **68.** $y = x(4^{-x})$

69. $g(x) = \log_3\sqrt{1-x}$ **70.** $h(x) = \log_5\frac{x}{x-1}$

In Exercises 71 and 72, find the indefinite integral.

71. $\int (x+1)5^{(x+1)^2}\,dx$ **72.** $\int \frac{2^{-1/t}}{t^2}\,dt$

73. ***Think About It*** Find the derivative of each function, given that a is constant.

(a) $y = x^a$ (b) $y = a^x$ (c) $y = x^x$ (d) $y = a^a$

74. ***Climb Rate*** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50\log_{10}\frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) Find the time when the altitude is increasing at the greatest rate.

75. ***Compound Interest*** How large a deposit, at 7 percent interest compounded continuously, must be made to obtain a balance of \$10,000 in 15 years?

76. ***Compound Interest*** A deposit earns interest at a rate of r percent compounded continuously and doubles in value in 10 years. Find r.

5.6

77. ***Air Pressure*** Under ideal conditions, air pressure decreases continuously with height above sea level at a rate proportional to the pressure at that height. If the barometer reads 30 inches at sea level and 15 inches at 18,000 feet, find the barometric pressure at 35,000 feet.

78. ***Radioactive Decay*** Radioactive radium has a half-life of approximately 1620 years. If the initial quantity is 5 grams, how much remains after 600 years?

79. ***Population Growth*** A population grows continuously at the rate of 1.5%. How long will it take the population to double?

80. ***Fuel Economy*** A certain automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12 percent for each 10 miles per hour.

(a) If s is the speed and y is the number of miles per gallon, find y as a function of s by solving the differential equation

$$\frac{dy}{ds} = -0.012y, \quad s > 50.$$

(b) Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per gallon					

5.7 **In Exercises 81–86, solve the differential equation.**

81. $\dfrac{dy}{dx} = \dfrac{x^2+3}{x}$

82. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1+e^{-2x}}$

83. $y' - 2xy = 0$

84. $y' - e^y \sin x = 0$

85. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$

86. $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$

87. Verify that the general solution $y = C_1x + C_2x^3$ satisfies the differential equation $x^2y'' - 3xy' + 3y = 0$. Then find the particular solution that satisfies the initial condition $y = 0$ and $y' = 4$ when $x = 2$.

88. ***Vertical Motion*** A falling object encounters air resistance that is proportional to its velocity. If the acceleration due to gravity is -9.8 meters per second per second, the net change in velocity is

$$\frac{dv}{dt} = kv - 9.8.$$

(a) Find the velocity of the object as a function of time if the initial velocity is v_0.

(b) Use the result in part (a) to find the limit of the velocity as t approaches infinity.

(c) Integrate the velocity function found in part (a) to find the position function s.

5.8 **In Exercises 89 and 90, sketch the graph of the function by hand.**

89. $f(x) = 2\arctan(x+3)$

90. $h(x) = -3\arcsin 2x$

In Exercises 91 and 92, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

91. (a) $\sin\left(\arcsin \frac{1}{2}\right)$
(b) $\cos\left(\arcsin \frac{1}{2}\right)$

92. (a) $\tan(\operatorname{arccot} 2)$
(b) $\cos\left(\operatorname{arcsec} \sqrt{5}\right)$

In Exercises 93–98, find the derivative of the function.

93. $y = \tan(\arcsin x)$

94. $y = \arctan(x^2 - 1)$

95. $y = x \operatorname{arcsec} x$

96. $y = \frac{1}{2}\arctan e^{2x}$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

98. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\dfrac{x}{2}, \quad 2 < x < 4$

5.9 **In Exercises 99–106, find the indefinite integral.**

99. $\displaystyle\int \frac{1}{e^{2x}+e^{-2x}}\,dx$

100. $\displaystyle\int \frac{1}{3+25x^2}\,dx$

101. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

102. $\displaystyle\int \frac{1}{16+x^2}\,dx$

103. $\displaystyle\int \frac{x}{16+x^2}\,dx$

104. $\displaystyle\int \frac{4-x}{\sqrt{4-x^2}}\,dx$

105. $\displaystyle\int \frac{\arctan(x/2)}{4+x^2}\,dx$

106. $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

107. ***Harmonic Motion*** A weight of mass m is attached to a spring and oscillates with simple harmonic motion. By Hooke's Law, you can determine that

$$\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}}\,dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y as a function of t, given that $y = 0$ when $t = 0$.

108. ***Think About It*** Sketch the region whose area is given by $\int_0^1 \arcsin x\,dx$. Then find the area of the region. Explain how you arrived at your answer.

5.10 **In Exercises 109 and 110, find the derivative of the function.**

109. $y = 2x - \cosh\sqrt{x}$

110. $y = x\tanh^{-1} 2x$

In Exercises 111 and 112, find the indefinite integral.

111. $\displaystyle\int \frac{x}{\sqrt{x^4-1}}\,dx$

112. $\displaystyle\int x^2 \operatorname{sech}^2 x^3\,dx$