We know that $L = 40$ and $k = 1$ from the form of the equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$. (*Note:* y is equivalent to P.) Solving for b in $y = \frac{L}{1 + be^{-kt}}$, we know that $b = \frac{L - y(0)}{y(0)} = \frac{40 - 8}{8} = 4$.

Therefore, $y = \frac{40}{1 + 4e^{-t}}$ is the final solution, by substitution.

AP Tips

- Homogeneous differential equations are *not* an AP topic.
- First-order linear differential equations are *not* an AP topic.

Past AP Free-Response Problems Covered by This Chapter

Note: These and other questions can be found at apcentral.com.

1998 BC 4
1999 BC 6b
2000 BC 6
2001 BC 5b, c
2002 BC 5
2004 BC 5
2005 AB 6, BC 4
2006 AB 5, BC 5
2007 AB 4b
2008 AB 5, BC 6
2009 BC 4a, c

Multiple-Choice Questions

A calculator may not be used for the following questions.

An asterisk (*) indicates the question is for BC students.

1. The general solution to the differential equation $\frac{dy}{dx} = y^2 \sin x$ is

   (A) $y = \sqrt[3]{3\cos x + C}$
   (B) $y = -\cos x + C$
   (C) $y = \sqrt[3]{\sin x + C}$
   (D) $y = \frac{1}{\cos x} + C$
   (E) $y = \sqrt[3]{-2\sec x + C}$

... $\frac{dy}{dx} = 2x$ and $y(1) = 2$, then the particular solution $y(x)$ is

   (A) $y = \ln(x^2) + 2$.
   (B) $y = \ln(x^2 + e^2 - 1)$.
   (C) $y = 2e^{x^2-1}$.
   (D) $y = x^2 + e^2 - 1$.
   (E) $y = \ln(x^2 + e - 4)$.

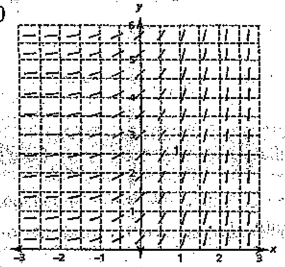
Questions 3–5 refer to the following information:

Consider the differential equation

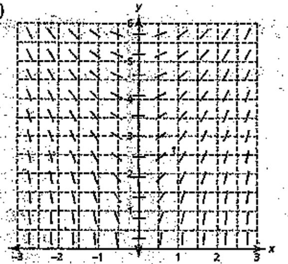
$\frac{dy}{dx} = \frac{4x}{y}$, for $y \geq 1$ only, with initial value $y(0) = 1$.

3. Which of the following is the slope field for the general solution to the given differential equation?

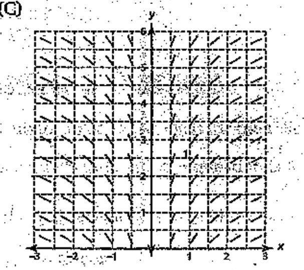
(A)



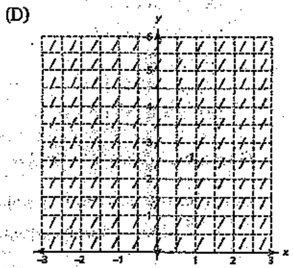
(B)



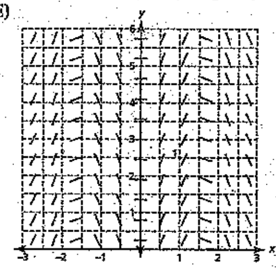
(C)



(D)



(E)



*4. Using Euler's Method with step size $\Delta x = 1/2$, what is the estimate for $y(1)$?
(A) 1
(B) 2
(C) $\sqrt{5}$
(D) 5/2
(E) 3

5. The particular solution $y(x)$ is
(A) $y = 2x$
(B) $y = \sqrt{4x^2 - 4}$
(C) $y = 2x^2 + 1$
(D) $y = e^{2x^2}$
(E) $y = \sqrt{4x^2 + 1}$

A calculator may be used for the following questions.

Questions 6–7 refer to the following information:

Water flows continuously from a large tank at a rate proportional to the amount of water remaining in the tank; that is, $\frac{dy}{dt} = ky$. There was initially 10,000 cubic feet of water in the tank, and at time $t = 4$ hours, 8000 cubic feet of water remained.

6. What is the value of k in the equation $\frac{dy}{dt} = ky$?
(A) −0.050
(B) −0.056
(C) −0.169
(D) −0.200
(E) −0.223

7. To the nearest cubic foot, how much water remained in the tank at time $t = 8$ hours?
(A) 5778
(B) 6000
(C) 6400
(D) 6458
(E) 6619

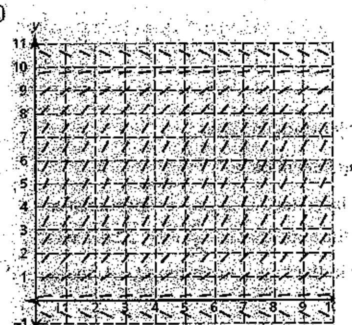
Questions 8–10 refer to the following information:

A population of rabbits in a certain habitat grows according to the differential equation $\frac{dy}{dt} = y\left(1 - \frac{1}{10}y\right)$ where t is measured in months ($t \geq 0$) and y is measured in hundreds of rabbits. There were initially 100 rabbits in this habitat; that is, $y(0) = 1$.

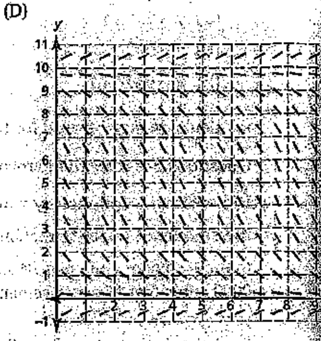
*8. What is the fastest growth rate, in rabbits per month, that this population exhibits?
(A) 50
(B) 100
(C) 200
(D) 250
(E) 500

*9. Which of the following slope fields represents an approximate general solution to the given differential equation?

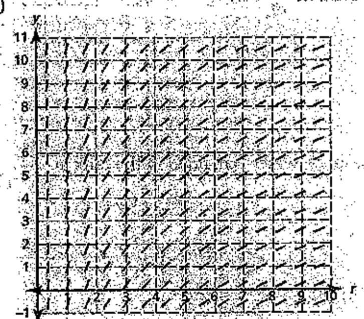
(A)



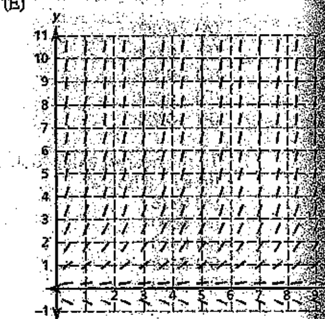
(D)



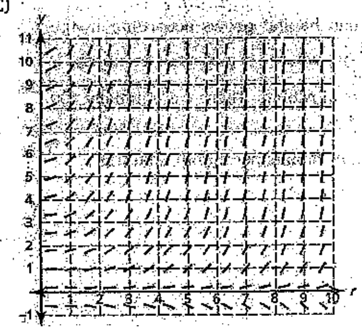
(B)



(E)



(C)



*10. Estimates of $y(t)$ can be produced using Euler's Method with step size $\Delta t = 1$. To the nearest rabbit, the estimate for $y(2)$ is
(A) 281
(B) 300
(C) 344
(D) 379
(E) 500

11. Water is being pumped continuously into a tank at a rate that is inversely proportional to the amount of water in the tank; that is, $\frac{dy}{dt} = \frac{k}{y}$, where y is the number of gallons of water in the tank after t minutes $(t \geq 0)$. Initially there were 5 gallons of water in the tank, and after 3 minutes there were 7 gallons. How many gallons of water were in the tank at time 18 minutes?
(A) $\sqrt{61}$
(B) $\sqrt{97}$
(C) 13
(D) $\sqrt{201}$
(E) 17

A calculator may not be used for the following questions.

Questions 12 and 13 refer to the following information:

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2} y \cos(x)$, for which the solution is $y = f(x)$. Let $f(0) = 2$.

12. Which of the following statements about the graph of $f(x)$ are true?
I. $f(x)$ has a vertical tangent when $y = 0$.
II. $f(x)$ has a horizontal tangent when $x = \frac{\pi}{2}$.
III. The slope of $f(x)$ at the point $(0,2)$ is 1.
(A) I only
(B) II only
(C) I and II only
(D) II and III only
(E) I, II, and III

13. The particular solution is
(A) $f(x) = x + 2$
(B) $f(x) = 2e^{-\frac{1}{2}\sin(x)}$
(C) $f(x) = \sqrt{\sin(x) + 4}$
(D) $f(x) = e^{\frac{1}{2}\sin(x)}$
(E) $f(x) = 2e^{\frac{1}{2}\sin(x)}$

Questions 14 and 15 refer to the following information:

Consider the differential equation $\frac{dy}{dx} = x + 2y$, for which the solution is $g(x)$.

14. Which of the following statements is true about the particular solution that contains $(0, -1)$?
(A) $g(x)$ is increasing and concave up.
(B) $g(x)$ is increasing and concave down.
(C) $g(x)$ is decreasing and concave up.
(D) $g(x)$ is decreasing and concave down.
(E) $g(x)$ is decreasing and linear.

*15. Let $y(x)$ be the particular solution that contains $(0,1)$. Using Euler's Method with step size $\Delta x = \frac{1}{2}$, what is the estimate for $y\left(\frac{1}{2}\right)$?
(A) $-\frac{1}{4}$
(B) $\frac{3}{4}$
(C) 1
(D) $\frac{3}{2}$
(E) 2

FREE-RESPONSE QUESTION

A calculator may not be used on this question.

1. A differentiable function $f(x)$ is defined such that, for all values of x in its domain, $f(x) = 3 + \int_0^{x^3} f\left(\sqrt[3]{t}\right) dt$.
   a. What is the domain of $f(x)$?
   b. For what value(s) of x is $f(x) = 3$?
   c. Show that $f'(x) = 3x^2 f(x)$.
   d. Solve the differential equation in (c) to find $f(x)$ in terms of x only.

Answers

MULTIPLE CHOICE

1. (D) Separating variables,
$$\int \frac{dy}{y^2} = \int \sin x\, dx \Rightarrow -\frac{1}{y} = -\cos x + C \Rightarrow y = \frac{1}{\cos x} + C = \sec x + C.$$