1. $\frac{dy}{dx} = \frac{7x^2}{y^3}$ $\quad y(3)=0$

$\int y^3\,dy = \int 7x^2\,dx$

$\frac{y^4}{4} = \frac{7x^3}{3} + C$

$0 = \frac{7(3)^3}{3} + C$

$-63 = C$

$\frac{y^4}{4} = \frac{7x^3}{3} - 63$

$\boxed{y^4 = \frac{28x^3}{3} - 252}$

2. $\frac{dy}{dx} = 5x^2 y$ $\quad (0,6)$

$\frac{dy}{y} = 5x^2\,dx$

$\ln y = \frac{5x^3}{3} + C$

$y = Ce^{5x^3/3}$

$6 = Ce^{5(0)^3/3}$

$6 = C$

$\boxed{y = 6e^{\frac{5x^3}{3}}}$

3. $\frac{dy}{dx} = \frac{1}{y + x^2 y}$ $\quad (0,2)$

$dy = \frac{1}{y(1+x^2)}\,dx$

$\int y\,dy = \int \frac{1}{1+x^2}\,dx$

$\frac{y^2}{2} = \arctan x + C$

$\frac{4}{2} = \arctan 0 + C$

$2 = C$

$\frac{y^2}{2} = \arctan x + 2$

$\boxed{y^2 = 2\arctan x + 4}$

4. $\frac{dy}{dx} = \frac{\sin x}{\cos y}$ $\quad y(0) = \frac{3\pi}{2}$

$\int \cos y\,dy = \int \sin x\,dx$

$\sin y = -\cos x + C$

$\sin \frac{3\pi}{2} = -\cos(0) + C$

$-1 = -1 + C$

$0 = C$

$\sin y = -\cos x$

$\boxed{y = \sin^{-1}(-\cos x)}$

5. $y = Ce^{kt}$

$y = 300e^{kt}$

$500 = 300e^{k(2)}$

$\frac{5}{3} = e^{2k}$

$\ln|5/3| = \ln e^{2k}$

$\ln 5/3 = 2k$

$\frac{\ln 5/3}{2} = k$

$y = 300e^{\frac{\ln 5/3}{2}(3)}$

$y = 645.497$

$\boxed{646 \text{ coyotes.}}$

6. $y = Ce^{kt}$

120?

$600 = Ce^{k(2)}$

$600 = Ce^{2k}$

$\frac{600}{e^{2k}} = C$

$75{,}000 = Ce^{8k}$

$75{,}000 = \frac{600}{e^{2k}}e^{8k}$

$75{,}000 = 600e^{6k}$

$k = .8047$

$y = Ce^{.8047t}$

$600 = Ce^{.8047(2)}$

$120 = C$

$\boxed{y = 120e^{.8047t}}$

10. $a = -9 m/s^2$

$v_0 = 18 m/s \quad s_0 = 45 m$

$s'' = a = -9$

$\frac{ds}{dt} = v = -9t + C$

$18 = -9(0) + C$

$v = -9t + 18$

$s = \frac{-9t^2}{2} + 18t + C$

$s = -\frac{9t^2}{2} + 18t + 45$

$s(4) = \frac{-9(4)^2}{2} + 18(4) + 45$

$= \boxed{45 \text{ feet}}$

(1a) $f'(x) = \frac{1+e^x}{x^2}$

$f'(3) = \frac{1+e^3}{3^2} = \frac{1+e^3}{9}$

$\boxed{y-6 = \frac{1+e^3}{9}(x-3)}$

$y = 6 + \frac{1+e^3}{9}(x-3)$

$f(3.1) = 6 + \frac{1+e^3}{9}(3.1-3)$

$= 6 + \frac{1+e^3}{9}(.1) = \boxed{6.234}$

(b) $x=3 \quad y=6$

x		y
3		6
3.05	$6 + .05\left(\frac{1+e^3}{3^2}\right)$	6.117
3.1	$6.117 + .05\left(\frac{1+e^{3.05}}{3.05^2}\right)$	$\boxed{6.236}$

$\boxed{f(3.1) \sim 6.236}$

$f''(x) = \frac{x^2(e^x) - (1+e^x)(2x)}{x^4}$

$= \frac{x^2e^x - 2x - 2xe^x}{x^4}$

$f''(3.1) = \frac{(3.1)^2 e^{3.1} - 2(3.1) - 2(3.1)e^{3.1}}{(3.1)^4}$

$= \frac{69.495}{(3.1)^4} > 0 \implies$ f(x) is concave up ∴ the tangent line is below the curve and an underestimate (c.) $\int_{3}^{3.1} \frac{1+e^x}{x^2}\,dx = .2377$

$f(3.1) = f(3) + .2377$

$\boxed{f(3.1) = 6.2377}$

(2a) $\frac{dw}{dt} = k(1200-w)$

$dw = k(1200-w)\,dt$

$\int \frac{dw}{1200-w} = \int k\,dt$

$u = 1200-w$

$du = -dw$

$-du = dw$

$-\ln|1200-w| = kt + C$

$\ln|1200-w| = -kt + C$

$1200-w = Ce^{-kt}$

$-w = Ce^{-kt} - 1200$

$w = -Ce^{-kt} + 1200$

$60 = -Ce^{-k(0)} + 1200$

$-1140 = -C$

$1140 = C$

$\boxed{w = -1140e^{-kt} + 1200}$

(b) $1100 = -1140e^{-k(3.05)} + 1200$

$-100 = -1140e^{-3.05k}$

$.0877 = e^{-3.05k}$

$\frac{\ln .0877}{-3.05} = k$

$\boxed{.798 = k}$

(c) $800 = -1140e^{-.798t} + 1200$

$-400 = -1140e^{-.798t}$

$.351 = e^{-.798t}$

$\frac{\ln .351}{-.798} = t$

$1.312 = t$

The cow was 1.3 years old.

(d) $\lim_{t\to\infty} -1140e^{-.798t} + 1200$

$= \boxed{1200 \text{ lbs}}$

(3) $\frac{dw}{dt} = kw$

$\int \frac{dw}{w} = \int k\,dt$

$\ln w = kt + C$

$w = Ce^{kt}$

$w = 10{,}000\,e^{kt}$

$8000 = 10{,}000\,e^{k(4)}$

$.8 = e^{4k}$

$-.056 = k$

$\boxed{w = 10{,}000\,e^{-.056t}}$

(4a) $\frac{dy}{dt} = ky$

$\frac{dy}{y} = k\,dt$

$\ln y = kt + C$

$y = Ce^{kt}$

$y = 1{,}000{,}000\,e^{kt}$

$500{,}000 = 1{,}000{,}000\,e^{k(6)}$

$\frac{1}{2} = e^{k(6)}$

$\ln \frac{1}{2} = 6k$

$-.116 = k$

$\boxed{y = 1{,}000{,}000\,e^{-.116t}}$

(c) $50{,}000 = 1{,}000{,}000\,e^{-.116t}$

$= e^{-.116t}$

$\frac{\ln .05}{-.116} = t$

$25.8 = t$

After about 25.8 years.

(b) $\frac{dy}{dt} = -.116(600{,}000)$

$\frac{dy}{dt} = -69314.718$

$\boxed{\text{decreasing @ 69,315 gal/yr}}$

(5a) $\frac{dv}{dt} = -2v - 32 \qquad v(0) = -50$

$dv = -2(v+16)\,dt$

$\int \frac{dv}{v+16} = \int -2\,dt$

$\ln|v+16| = -2t + C$

$v + 16 = Ce^{-2t}$

$-50 + 16 = Ce^{-2(0)}$

$-34 = C$

$\boxed{v = -34e^{-2t} - 16}$

(b) $\lim_{t \to \infty} -34e^{-2t} - 16$

-16 ft/sec

(c) $-20 = -34e^{-2t} - 16$

$-4 = -34e^{-2t}$

$.118 = e^{-2t}$

$1.07 = t$

after about 1 second

1987 BC1

At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $y' = ky$, where k is a constant and y is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

(a) Write an expression for y at any time $t \geq 0$.

(b) By what factor will the population have increased in the first 10 days?

(c) At what time t, in days, will the population have increased by a factor of 6?

1987 BC1
Solution

(a) $y = Ae^{kt}$

$y(0) = 1000 = A$

$3 = e^{k \cdot 5}$

$k = \dfrac{\ln 3}{5}$

Therefore $y = 1000e^{\frac{t \ln 3}{5}}$ or $y = 1000 \cdot 3^{t/5}$

(b) $y(10) = 1000e^{\frac{10 \ln 3}{5}} = 1000 \cdot 3^2$

Therefore the population will have increased by a factor of 9.

(c) $6000 = 1000e^{\frac{t \ln 3}{5}}$

$6 = e^{\frac{t \ln 3}{5}}$

$t = \dfrac{5 \ln 6}{\ln 3}$

2

1993 AB6

Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 700$, find k.

(c) Find $\lim_{t \to \infty} P(t)$.

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a)
$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2$$
$$= 2y^3(6 - 2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ \quad <-2> \text{ product rule or chain rule error} \\ 1 : \text{value at } \left(3, \frac{1}{4}\right) \end{cases}$

(b)
$$\frac{1}{y^2}\,dy = (6 - 2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$
$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

6 : $\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = \frac{1}{4} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a) $$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2$$
$$= 2y^3(6 - 2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

$$3: \begin{cases} 2: \dfrac{d^2y}{dx^2} \\ \quad <-2> \text{ product rule or chain rule error} \\ 1: \text{value at } \left(3, \frac{1}{4}\right) \end{cases}$$

(b) $$\frac{1}{y^2}\,dy = (6 - 2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

$$6: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivative of } dy \text{ term} \\ 1: \text{antiderivative of } dx \text{ term} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition } f(3) = \frac{1}{4} \\ 1: \text{solves for } y \end{cases}$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

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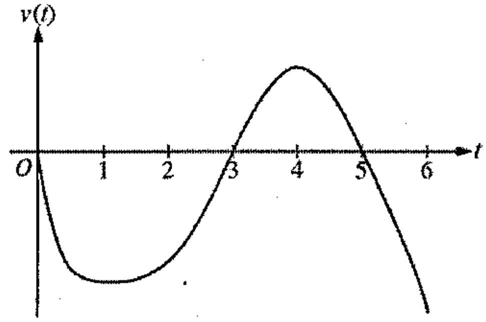
Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

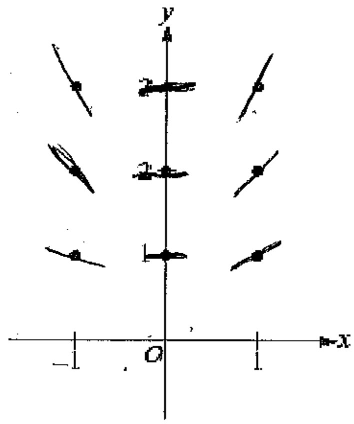
(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t)\,dt \\ 1 : \text{conclusion} \end{cases}$
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	$3 : \begin{cases} 1 : \text{positions at } t = 3,\ t = 5, \\ \quad \text{and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

5. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

(b)

x		y
0		3
.1	$3 + .1\left(\frac{3(0)}{2}\right)$	3
.2	$3 + .1\left(\frac{3(.1)}{2}\right)$	3.015

$\boxed{f(0.2) \approx 3.015}$

(c) $\frac{dy}{dx} = \frac{xy}{2}$

$\int \frac{dy}{y} = \int \frac{1}{2} x \, dx$

$\ln y = x^2/4 + C$

$y = Ce^{x^2/4}$

$3 = Ce^{0^2/4}$

$3 = C$

$y = 3e^{x^2/4}$

$f(0.2) = 3e^{(.2)^2/4}$

$\boxed{f(0.2) = 3e^{.01} \approx 3.030}$