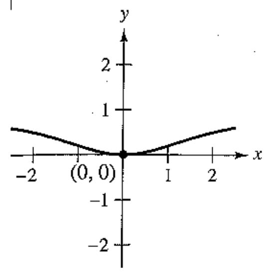
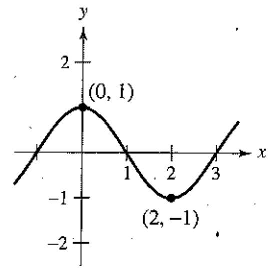
EXERCISES FOR SECTION 3.1

In Exercises 1–6, find the value of the derivative (if it exists) at each indicated extremum.

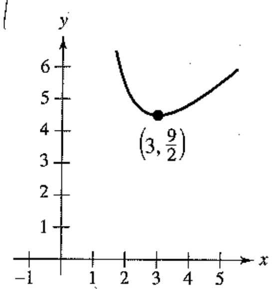
1. $f(x) = \dfrac{x^2}{x^2 + 4}$



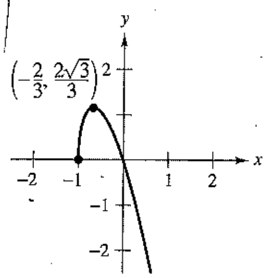
2. $f(x) = \cos \dfrac{\pi x}{2}$



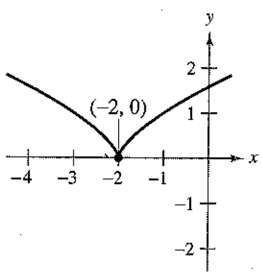
3. $f(x) = x + \dfrac{27}{2x^2}$



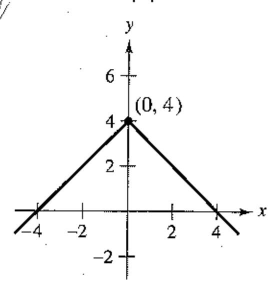
4. $f(x) = -3x\sqrt{x+1}$



5. $f(x) = (x + 2)^{2/3}$

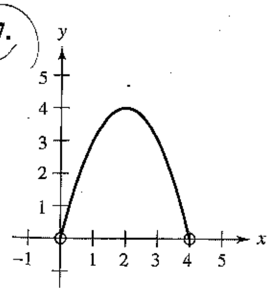


6. $f(x) = 4 - |x|$



In Exercises 7–10, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, relative minimum, absolute maximum, absolute minimum, or none of these at each critical number on the interval shown.

7.



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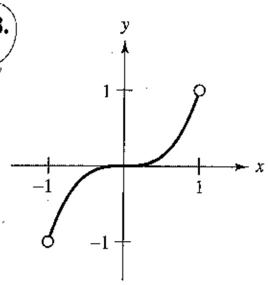

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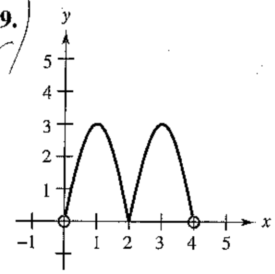
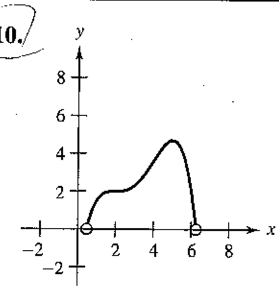


10.



In Exercises 11–16, find any critical numbers of the function.

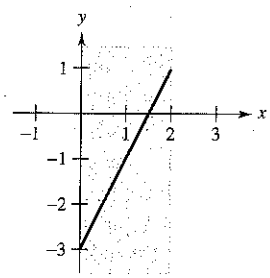
11. $f(x) = x^2(x - 3)$
12. $g(x) = x^2(x^2 - 4)$
13. $g(t) = t\sqrt{4 - t},\ t < 3$
14. $f(x) = \dfrac{4x}{x^2 + 1}$
15. $h(x) = \sin^2 x + \cos x$, $0 < x < 2\pi$
16. $f(\theta) = 2 \sec \theta + \tan \theta$, $0 < \theta < 2\pi$

In Exercises 17–32, locate the absolute extrema of the function on the closed interval.

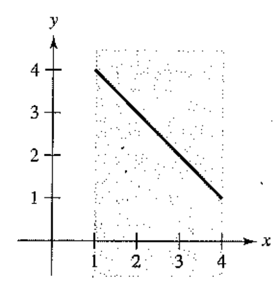
17. $f(x) = 2(3 - x),\ [-1, 2]$
18. $f(x) = \dfrac{2x + 5}{3},\ [0, 5]$
19. $f(x) = -x^2 + 3x,\ [0, 3]$
20. $f(x) = x^2 + 2x - 4,\ [-1, 1]$
21. $f(x) = x^3 - \dfrac{3}{2}x^2,\ [-1, 2]$
22. $f(x) = x^3 - 12x,\ [0, 4]$
23. $y = 3x^{2/3} - 2x,\ [-1, 1]$
24. $g(x) = \sqrt[3]{x},\ [-1, 1]$
25. $g(t) = \dfrac{t^2}{t^2 + 3},\ [-1, 1]$
26. $y = 3 - |t - 3|,\ [-1, 5]$
27. $h(s) = \dfrac{1}{s - 2},\ [0, 1]$
28. $h(t) = \dfrac{t}{t - 2},\ [3, 5]$
29. $f(x) = \cos \pi x,\ \left[0, \dfrac{1}{6}\right]$
30. $g(x) = \sec x,\ \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$
31. $y = \dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right),\ [1, 2]$
32. $y = x^2 - 2 - \cos x,\ [-1, 3]$

In Exercises 33–36, locate the absolute extrema of the function (if any exist) over the indicated intervals.

33. $f(x) = 2x - 3$
(a) $[0, 2]$ (b) $[0, 2)$
(c) $(0, 2]$ (d) $(0, 2)$



34. $f(x) = 5 - x$
(a) $[1, 4]$ (b) $[1, 4)$
(c) $(1, 4]$ (d) $(1, 4)$



35. $f(x) = x^2 - 2x$
(a) $[-1, 2]$ (b) $(1, 3]$
(c) $(0, 2)$ (d) $[1, 4)$

36. $f(x) = \sqrt{4 - x^2}$
(a) $[-2, 2]$ (b) $[-2, 0)$
(c) $(-2, 2)$ (d) $[1, 2)$

In Exercises 37–40, use a graphing utility to graph the function. Locate the absolute extrema of the function on the closed interval.

Function	*Interval*
37. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$	$[0, 3]$
38. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$	$[1, 5]$
39. $f(x) = \dfrac{3}{x - 1}$	$(1, 4]$
40. $f(x) = \dfrac{2}{2 - x}$	$[0, 2)$

In Exercises 41 and 42, (a) use a computer algebra system to graph the function and approximate any absolute extrema on the indicated interval. (b) Use the utility to find any critical numbers, and use them to find any absolute extrema not located at the endpoints. Compare the results with those in part (a).

Function	*Interval*
41. $f(x) = 3.2x^5 + 5x^3 - 3.5x$	$[0, 1]$
42. $f(x) = \dfrac{4}{3}x\sqrt{3 - x}$	$[0, 3]$

In Exercises 43 and 44, use a computer algebra system to find the maximum value of $|f''(x)|$ on the closed interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 4.6.)

Function	*Interval*
43. $f(x) = \sqrt{1 + x^3}$	$[0, 2]$
44. $f(x) = \dfrac{1}{x^2 + 1}$	$\left[\dfrac{1}{2}, 3\right]$

In Exercises 45 and 46, use a computer algebra system to find the maximum value of $|f^{4}(x)|$ on the closed interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 4.6.)

Function	*Interval*
45. $f(x) = (x + 1)^{2/3}$	$[0, 2]$
46. $f(x) = \dfrac{1}{x^2 + 1}$	$[-1, 1]$

47. Explain why the function $f(x) = \tan x$ has a maximum on $[0, \pi/4]$ but not on $[0, \pi]$.

48. ***Writing*** Write a short paragraph explaining why a continuous function on an open interval may not have a maximum or minimum. Illustrate your explanation with a sketch of the graph of a function.

Getting at the Concept

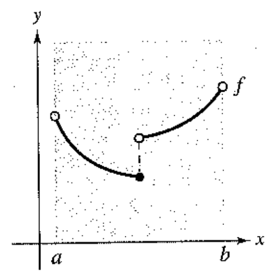
In Exercises 49 and 50, graph a function on the interval $[-2, 5]$ having the given characteristics.

49. Absolute maximum at $x = -2$
Absolute minimum at $x = 1$
Relative maximum at $x = 3$

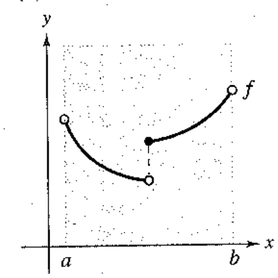
50. Relative minimum at $x = -1$
Critical number at $x = 0$, but no extrema
Absolute maximum at $x = 2$
Absolute minimum at $x = 5$

In Exercises 51–54, determine from the graph whether f has a minimum in the open interval (a, b).

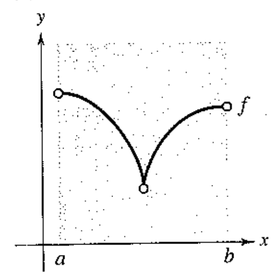
51. (a)



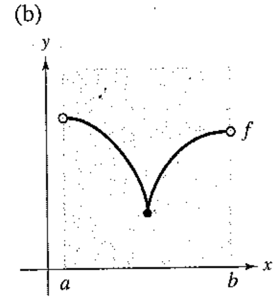
(b)



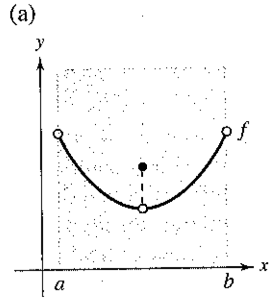
52. (a)



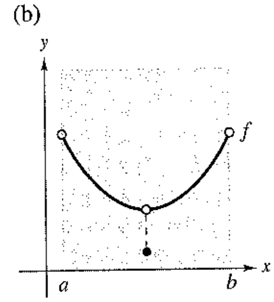
(b)



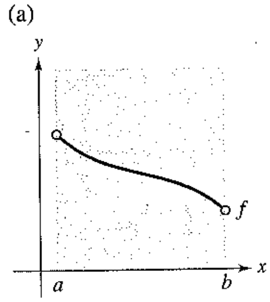
53. (a)



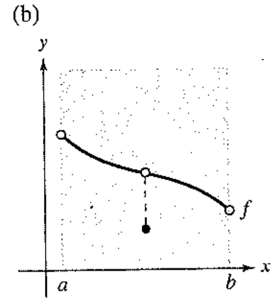
(b)



54. (a)



(b)

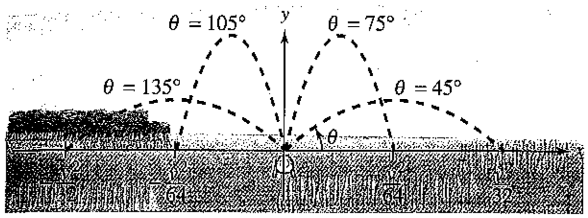


55. ***Power*** The formula for the power output P of a battery is $P = VI - RI^2$ where V is the electromotive force in volts, R is the resistance, and I is the current. Find the current (measured in amperes) that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohm. Assume that a 15-ampere fuse bounds the output in the interval $0 \le I \le 15$. Could the power output be increased by replacing the 15-ampere fuse with a 20-ampere fuse? Explain.

56. ***Lawn Sprinkler*** A lawn sprinkler is constructed in such a way that $d\theta/dt$ is constant, where θ ranges between 45° and 135° (see figure). The distance the water travels horizontally is

$$x = \frac{v^2 \sin 2\theta}{32}, \qquad 45° \le \theta \le 135°$$

where v is the speed of the water. Find dx/dt and explain why this lawn sprinkler does not water evenly. What part of the lawn receives the most water?



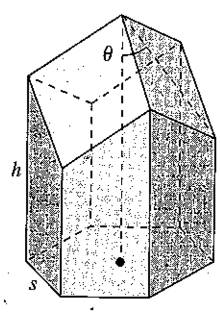
Water sprinkler: $45° \le \theta \le 135°$

FOR FURTHER INFORMATION For more information on the "calculus of lawn sprinklers," see the article "Design of an Oscillating Sprinkler" by Bart Braden in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

57. ***Honeycomb*** The surface area of a cell in a honeycomb is

$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos \theta}{\sin \theta}\right)$$

where h and s are positive constants and θ is the angle at which the upper faces meet the altitude of the cell. Find the angle $\theta\ (\pi/6 \le \theta \le \pi/2)$ that minimizes the surface area S.



FOR FURTHER INFORMATION For more information on the geometric structure of a honeycomb cell, see the article "The Design of Honeycombs" by Anthony L. Peressini in UMAP Module 502, published by COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA. To view this article, go to the website *www.matharticles.com*.

58. ***Inventory Cost*** A retailer has determined that the cost C of ordering and storing x units of a certain product is

$$C = 2x + \frac{300{,}000}{x}, \qquad 1 \le x \le 300.$$

The delivery truck can bring at most 300 units per order. Find the order size that will minimize cost. Could the cost be decreased if the truck were replaced with one that could bring at most 400 units? Explain.

59. ***Highway Design*** In order to build a highway it is necessary to fill a section of a valley where the grades (slopes) of the sides are 9% and 6% (see figure). The top of the filled region will have the shape of a parabolic arc that is tangent to the two slopes at the points A and B. The horizontal distance between the points A and B is 1000 feet.

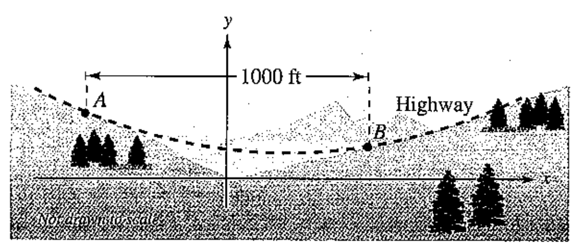
(a) Find a quadratic function $y = ax^2 + bx + c$, $-500 \le x \le 500$, that describes the top of the filled region.

(b) Complete the table giving the depths d of the fill at the specified values of x.

x	-500	-400	-300	-200	-100
d					

x	0	100	200	300	400	500
d						

(c) What will be the lowest point on the completed highway? Will it be directly over the point where the two hillsides come together?



60. Find all critical numbers of the greatest integer function $f(x) = [\![x]\!]$.

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

62. If a function is continuous on a closed interval, then it must have a minimum on the interval.

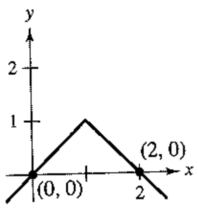
63. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

64. If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x - k)$, where k is a constant.

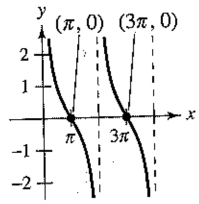
EXERCISES FOR SECTION 3.2

In Exercises 1 and 2, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. $f(x) = 1 - |x - 1|$



2. $f(x) = \cot \frac{x}{2}$



In Exercises 3–6, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two intercepts.

3. $f(x) = x^2 - x - 2$
4. $f(x) = x(x - 3)$
5. $f(x) = x\sqrt{x + 4}$
6. $f(x) = -3x\sqrt{x + 1}$

In Exercises 7–20, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

7. $f(x) = x^2 - 2x, [0, 2]$
8. $f(x) = x^2 - 5x + 4, [1, 4]$
9. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$
10. $f(x) = (x - 3)(x + 1)^2, [-1, 3]$
11. $f(x) = x^{2/3} - 1, [-8, 8]$
12. $f(x) = 3 - |x - 3|, [0, 6]$
13. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$
14. $f(x) = \dfrac{x^2 - 1}{x}, [-1, 1]$
15. $f(x) = \sin x, [0, 2\pi]$
16. $f(x) = \cos x, [0, 2\pi]$
17. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x, \left[0, \dfrac{\pi}{6}\right]$
18. $f(x) = \cos 2x, \left[-\dfrac{\pi}{12}, \dfrac{\pi}{6}\right]$
19. $f(x) = \tan x, [0, \pi]$
20. $f(x) = \sec x, \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

In Exercises 21–24, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

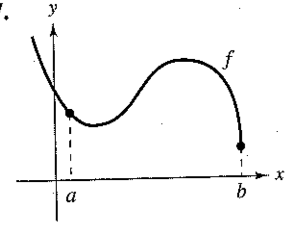
21. $f(x) = |x| - 1, [-1, 1]$
22. $f(x) = x - x^{1/3}, [0, 1]$
23. $f(x) = 4x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$
24. $f(x) = \dfrac{x}{2} - \sin \dfrac{\pi x}{6}, [-1, 0]$

25. *Vertical Motion* The height of a ball t seconds after it is thrown upward from a height of 32 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 32$.
    (a) Verify that $f(1) = f(2)$.
    (b) According to Rolle's Theorem, what must be the velocity at some time in the interval $(1, 2)$? Find that time.

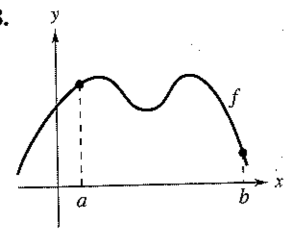
26. *Reorder Costs* The ordering and transportation cost C of components used in a manufacturing process is approximated by
$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$$
where C is measured in thousands of dollars and x is the order size in hundreds.
    (a) Verify that $C(3) = C(6)$.
    (b) According to Rolle's Theorem, the rate of change of cost must be 0 for some order size in the interval $(3, 6)$. Find that order size.

In Exercises 27 and 28, copy the graph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

27.



28.



In Exercises 29 and 30, explain why the Mean Value Theorem does not apply to the function on the interval [0, 6].

29. $f(x) = \dfrac{1}{x - 3}$
30. $f(x) = |x - 3|$

In Exercises 31–38, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

31. $f(x) = x^2, [-2, 1]$
32. $f(x) = x(x^2 - x - 2), [-1, 1]$
33. $f(x) = x^{2/3}, [0, 1]$
34. $f(x) = \dfrac{x + 1}{x}, \left[\frac{1}{2}, 2\right]$
35. $f(x) = \sqrt{2 - x}, [-7, 2]$
36. $f(x) = x^3, [0, 1]$
37. $f(x) = \sin x, [0, \pi]$
38. $f(x) = 2\sin x + \sin 2x, [0, \pi]$

In Exercises 39–42, use a graphing utility to (a) graph the function f on the indicated interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the indicated interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

39. $f(x) = \dfrac{x}{x + 1}, \left[-\frac{1}{2}, 2\right]$
40. $f(x) = x - 2\sin x, [-\pi, \pi]$
41. $f(x) = \sqrt{x}, [1, 9]$
42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5, [0, 5]$

43. *Vertical Motion* The height of an object t seconds after it is dropped from a height of 500 meters is $s(t) = -4.9t^2 + 500$.
   (a) Find the average velocity of the object during the first 3 seconds.
   (b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall the instantaneous velocity equals the average velocity. Find that time.

44. *Sales* A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

where t is the time in months.
   (a) Find the average value of $S(t)$ during the first year.
   (b) During what month does $S'(t)$ equal the average value during the first year?

Getting at the Concept

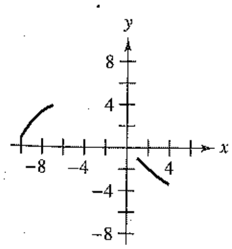
45. Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

46. Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).
   (a) $g(x) = f(x) + k$ (b) $g(x) = f(x - k)$
   (c) $g(x) = f(kx)$

47. A plane begins its takeoff at 2:00 P.M. on a 2500-mile flight. The plane arrives at its destination at 7:30 P.M. Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

48. When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

49. *Graphical Reasoning* The figure gives two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



   (a) Explain why f must have at least one zero in $[-10, 4]$.
   (b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?
   (c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.
   (d) Make a possible sketch of the function with two zeros of f' on the interval $[-10, 4]$.
   (e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

50. Consider the function $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$.
   (a) Use a graphing utility to graph f and f'.
   (b) Is f a continuous function? Is f' a continuous function?
   (c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.
   (d) Evaluate, if possible, $\lim_{x\to 3^-} f'(x)$ and $\lim_{x\to 3^+} f'(x)$.

***Think About It* In Exercises 51 and 52, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.**

51. f is continuous on $[-5, 5]$.

52. f is not continuous on $[-5, 5]$.

***True or False?* In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. The Mean Value Theorem can be applied to $f(x) = 1/x$ on the interval $[-1, 1]$.

54. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

55. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

56. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.

57. Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

58. Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

59. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

60. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.

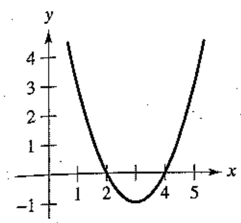
61. Use the result of Exercise 60 to show that $f(x) = \frac{1}{2}\cos x$ has at most one fixed point.

62. Prove that $|\cos x - \cos y| \le |x - y|$ for all x and y.

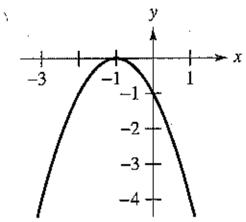
EXERCISES FOR SECTION 3.3

In Exercises 1–10, identify the open intervals on which the function is increasing or decreasing.

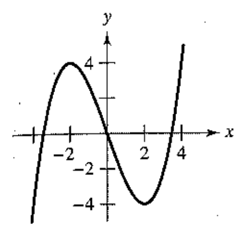
1. $f(x) = x^2 - 6x + 8$



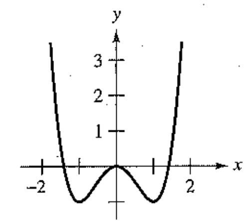
2. $y = -(x + 1)^2$



3. $y = \dfrac{x^3}{4} - 3x$



4. $f(x) = x^4 - 2x^2$



5. $f(x) = \dfrac{1}{x^2}$

6. $y = \dfrac{x^2}{x + 1}$

7. $g(x) = x^2 - 2x - 8$

8. $h(x) = 27x - x^3$

9. $y = x\sqrt{16 - x^2}$

10. $y = x + \dfrac{4}{x}$

In Exercises 11–32, find the critical numbers of f (if any). Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. Use a graphing utility to confirm your results.

11. $f(x) = x^2 - 6x$

12. $f(x) = x^2 + 8x + 10$

13. $f(x) = -2x^2 + 4x + 3$

14. $f(x) = -(x^2 + 8x + 12)$

15. $f(x) = 2x^3 + 3x^2 - 12x$

16. $f(x) = x^3 - 6x^2 + 15$

17. $f(x) = x^2(3 - x)$

18. $f(x) = (x + 2)^2(x - 1)$

19. $f(x) = \dfrac{x^5 - 5x}{5}$

20. $f(x) = x^4 - 32x + 4$

21. $f(x) = x^{1/3} + 1$

22. $f(x) = x^{2/3} - 4$

23. $f(x) = (x - 1)^{2/3}$

24. $f(x) = (x - 1)^{1/3}$

25. $f(x) = 5 - |x - 5|$

26. $f(x) = |x + 3| - 1$

27. $f(x) = x + \dfrac{1}{x}$

28. $f(x) = \dfrac{x}{x + 1}$

29. $f(x) = \dfrac{x^2}{x^2 - 9}$

30. $f(x) = \dfrac{x + 3}{x^2}$

31. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

32. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

In Exercises 33–36, consider the function on the interval $(0, 2\pi)$. Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. Use a graphing utility to confirm your results.

33. $f(x) = \dfrac{x}{2} + \cos x$

34. $f(x) = \sin x \cos x$

35. $f(x) = \sin^2 x + \sin x$

36. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

In Exercises 37–40, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of f and f' on the same set of coordinate axes over the indicated interval, (c) find the critical numbers of f in the open interval, and (d) find the interval(s) on which f' is positive and the interval(s) on which it is negative. Compare the behavior of f and the sign of f'.

37. $f(x) = 2x\sqrt{9 - x^2}$, $[-3, 3]$

38. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right)$, $[0, 5]$

39. $f(t) = t^2 \sin t$, $[0, 2\pi]$

40. $f(x) = \dfrac{x}{2} + \cos \dfrac{x}{2}$, $[0, 4\pi]$

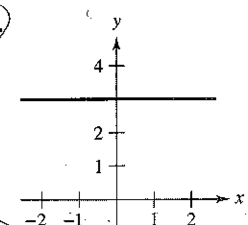
In Exercises 41 and 42, use symmetry, extrema, and zeros to sketch the graph of f. How do the functions f and g differ? Explain.

41. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1}$, $g(x) = x(x^2 - 3)$

42. $f(t) = \cos^2 t - \sin^2 t$, $g(t) = 1 - 2\sin^2 t$, $(-2, 2)$

Think About It **In Exercises 43–48, the graph of f is shown in the figure. Sketch a graph of the derivative of f. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

43.



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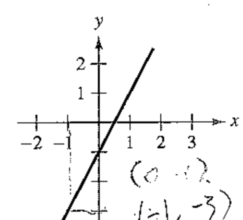

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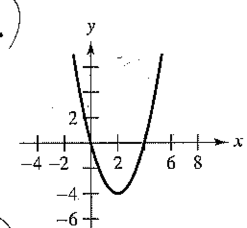

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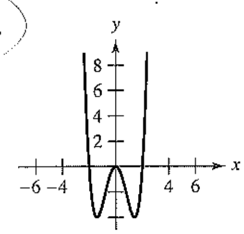

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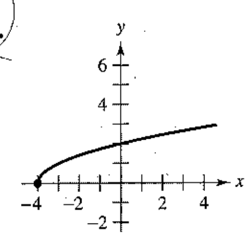

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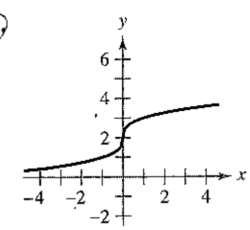


Getting at the Concept

In Exercises 49–54, assume that f is differentiable for all x. The sign of f' is as follows.

$f'(x) > 0$ on $(-\infty, -4)$

$f'(x) < 0$ on $(-4, 6)$

$f'(x) > 0$ on $(6, \infty)$

Supply the appropriate inequality for the indicated value of c.

	Function	*Sign of $g'(c)$*
49.	$g(x) = f(x) + 5$	$g'(0)$ ▢ 0
50.	$g(x) = 3f(x) - 3$	$g'(-5)$ ▢ 0
51.	$g(x) = -f(x)$	$g'(-6)$ ▢ 0
52.	$g(x) = -f(x)$	$g'(0)$ ▢ 0
53.	$g(x) = f(x - 10)$	$g'(0)$ ▢ 0
54.	$g(x) = f(x - 10)$	$g'(8)$ ▢ 0

55. Sketch the graph of an arbitrary function f such that

$$f'(x)\begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4 \\ < 0, & x > 4. \end{cases}$$

56. A differentiable function f has one critical number at $x = 5$. Identify the relative extrema of f at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

57. ***Think About It*** The function f is differentiable on the interval $[-1, 1]$. The table shows the values of f' for selected values of x. Sketch the graph of f, approximate the critical numbers, and identify the relative extrema.

x	-1	-0.75	-0.50	-0.25
$f'(x)$	-10	-3.2	-0.5	0.8

x	0	0.25	0.50	0.75	1
$f'(x)$	5.6	3.6	-0.2	-6.7	-20.1

58. ***Rolling a Ball Bearing*** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is θ. The distance (in meters) the ball bearing rolls in t seconds is

$$s(t) = 4.9(\sin\theta)t^2.$$

(a) Determine the speed of the ball bearing after t seconds.

(b) What value of θ will produce the maximum speed at a particular time?

59. ***Numerical, Graphical, and Analytic Analysis*** Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

x	0.5	1	1.5	2	2.5	3
$f(x)$						
$g(x)$						

(b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint:* Show that $h'(x) > 0$ where $h = f - g$.]

60. ***Numerical, Graphical, and Analytic Analysis*** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

(a) Complete the table and use the table to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$							

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

(c) Use calculus to determine analytically the time when the concentration is greatest.

61. ***Trachea Contraction*** Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. Suppose the velocity of the air during coughing is

$$v = k(R - r)r^2, \quad 0 \leq r < R$$

where k is constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

62. ***Profit*** The profit P (in dollars) made by a fast-food restaurant selling x hamburgers is

$$P = 2.44x - \frac{x^2}{20{,}000} - 5000, \quad 0 \leq x \leq 35{,}000.$$

Find the open intervals on which P is increasing or decreasing.

63. ***Power*** The electric power P in watts in a direct-current circuit with two resistors R_1 and R_2 connected in series is

$$P = \frac{vR_1R_2}{(R_1 + R_2)^2}$$

where v is the voltage. If v and R_1 are held constant, what resistance R_2 produces maximum power?

64. ***Electrical Resistance*** The resistance R of a certain type of resistor is

$$R = \sqrt{0.001T^4 - 4T + 100}$$

where R is measured in ohms and the temperature T is measured in degrees Celsius.

(a) Use a computer algebra system to find dR/dT and the critical number of the function. Determine the minimum resistance for this type of resistor.

(b) Use a graphing utility to graph the function R and use the graph to approximate the minimum resistance for this type of resistor.

65. ***Modeling Data*** The number of bankruptcies (in thousands) for the years 1981 through 1998 are as follows.

1981: 360.3; 1982: 367.9; 1983: 374.7; 1984: 344.3;
1985: 364.5; 1986: 477.9; 1987: 561.3; 1988: 594.6;
1989: 643.0; 1990: 725.5; 1991: 880.4; 1992: 972.5;
1993: 918.7; 1994: 845.3; 1995: 858.1; 1996: 1042.1;
1997: 1317.0; 1998: 1411.4

(Source: Administrative Office of the U.S. Courts)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$B = at^4 + bt^3 + ct^2 + dt + e$$

for the data. (Let $t = 1$ represent 1981.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Analytically find the minimum of the model and compare the result with the actual data.

66. Use a graphing utility to graph $f(x) = 2 \sin 3x + 4 \cos 3x$. Find the maximum value of f. How could you use calculus to estimate the maximum?

Creating Polynomial Functions **In Exercises 67–70, find a polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the x-coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.

67. Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$
68. Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$
69. Relative minima: $(0, 0)$, $(4, 0)$
Relative maximum: $(2, 4)$
70. Relative minimum: $(1, 2)$
Relative maxima: $(-1, 4)$, $(3, 4)$

True or False? **In Exercises 71–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. The sum of two increasing functions is increasing.
72. The product of two increasing functions is increasing.
73. Every nth-degree polynomial has $(n - 1)$ critical numbers.
74. An nth-degree polynomial has at most $(n - 1)$ critical numbers.
75. There is a relative maximum or minimum at each critical number.
76. The relative maxima of the function f are $f(1) = 4$ and $f(3) = 10$. Therefore, f has at least one minimum for some x in the interval $(1, 3)$.

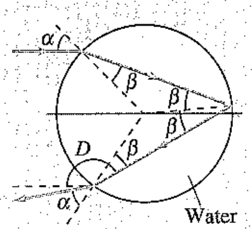
77. Prove the second case of Theorem 3.5.
78. Prove the second case of Theorem 3.6.
79. Let $x > 0$ and $n > 1$ be real numbers. Prove that $(1 + x)^n > 1 + nx$.

SECTION PROJECT RAINBOWS

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that $(\sin \alpha)/(\sin \beta) = k$, where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.

(a) Sketch the graph of D for $0 \le \alpha \le \pi/2$. Use a graphing utility with

$$D = \pi + 2\alpha - 4 \sin^{-1}\left(\frac{1}{k} \sin \alpha\right).$$



(b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

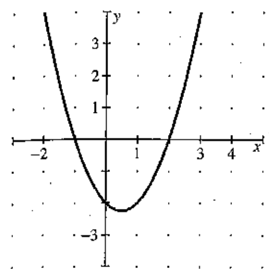
For water, what is the minimum angle of deflection, D_{min}? (The angle $\pi - D_{min}$ is called the *rainbow angle*.) What value of α produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, α, is called a *rainbow ray*.)

FOR FURTHER INFORMATION For more information about the mathematics of rainbows, see the article "Somewhere Within the Rainbow" by Steven Janke in *The UMAP Journal*. To view this article, go to the website *www.matharticles.com*.

EXERCISES FOR SECTION 3.4

In Exercises 1–10, determine the open intervals on which the graph is concave upward or concave downward.

1. $y = x^2 - x - 2$



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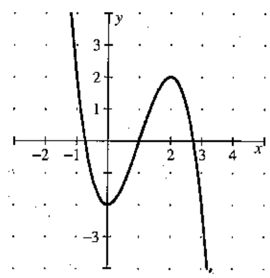
2. $y = -x^3 + 3x^2 - 2$



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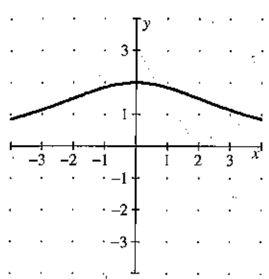
3. $f(x) = \dfrac{24}{x^2 + 12}$



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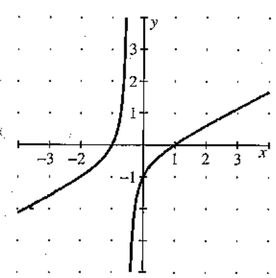
4. $f(x) = \dfrac{x^2 - 1}{2x + 1}$



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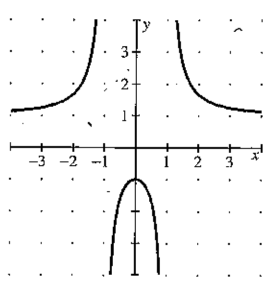
5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$



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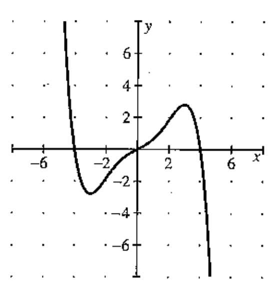
6. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$



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7. $g(x) = 3x^2 - x^3$

8. $h(x) = x^5 - 5x + 2$

9. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

10. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

In Exercises 11–26, find the points of inflection and discuss the concavity of the graph of the function.

11. $f(x) = x^3 - 6x^2 + 12x$
12. $f(x) = 2x^3 - 3x^2 - 12x + 5$
13. $f(x) = \frac{1}{4}x^4 - 2x^2$
14. $f(x) = 2x^4 - 8x + 3$
15. $f(x) = x(x - 4)^3$
16. $f(x) = x^3(x - 4)$
17. $f(x) = x\sqrt{x + 3}$
18. $f(x) = x\sqrt{x + 1}$
19. $f(x) = \dfrac{x}{x^2 + 1}$
20. $f(x) = \dfrac{x + 1}{\sqrt{x}}$
21. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$
22. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$
23. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$
24. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$
25. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$
26. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

In Exercises 27–40, find all relative extrema. Use the Second Derivative Test where applicable.

27. $f(x) = x^4 - 4x^3 + 2$
28. $f(x) = x^2 + 3x - 8$
29. $f(x) = (x - 5)^2$
30. $f(x) = -(x - 5)^2$
31. $f(x) = x^3 - 3x^2 + 3$
32. $f(x) = x^3 - 9x^2 + 27x$
33. $g(x) = x^2(6 - x)^3$
34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$
35. $f(x) = x^{2/3} - 3$
36. $f(x) = \sqrt{x^2 + 1}$
37. $f(x) = x + \dfrac{4}{x}$
38. $f(x) = \dfrac{x}{x - 1}$
39. $f(x) = \cos x - x, \quad [0, 4\pi]$
40. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

In Exercises 41–44, use a computer algebra system to analyze the function over the indicated interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

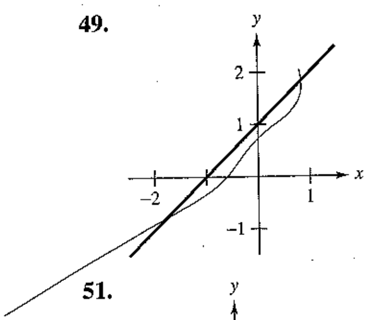
41. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$
42. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$
43. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$
44. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

Getting at the Concept

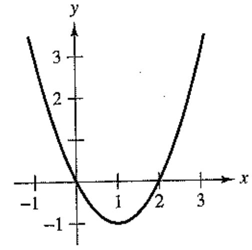
45. Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
46. Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
47. Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.
48. S represents weekly sales of a product. What can be said of S' and S'' for each of the following?
    (a) The rate of change of sales is increasing.
    (b) Sales are increasing at a slower rate.
    (c) The rate of change of sales is constant.
    (d) Sales are steady.
    (e) Sales are declining, but at a slower rate.
    (f) Sales have bottomed out and have started to rise.

In Exercises 49–52, graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

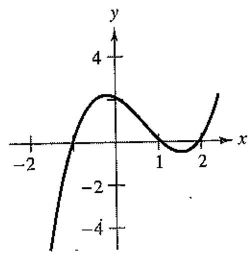
49.



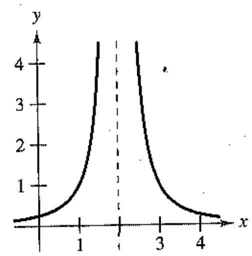
50.



51.



52.



Think About It **In Exercises 53–56, sketch the graph of a function f having the indicated characteristics.**

53. $f(2) = f(4) = 0$
$f(3)$ is defined.
$f'(x) < 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ if $x > 3$
$f''(x) < 0, x \neq 3$

54. $f(0) = f(2) = 0$
$f'(x) > 0$ if $x < 1$
$f'(1) = 0$
$f'(x) < 0$ if $x > 1$
$f''(x) < 0$

55. $f(2) = f(4) = 0$
$f'(x) > 0$ if $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ if $x > 3$
$f''(x) > 0, x \neq 3$

56. $f(0) = f(2) = 0$
$f'(x) < 0$ if $x < 1$
$f'(1) = 0$
$f'(x) > 0$ if $x > 1$
$f''(x) > 0$

57. ***Think About It*** The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.)

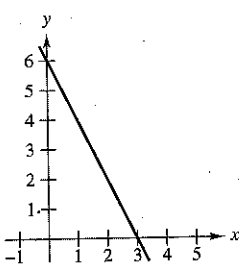


Figure for 57

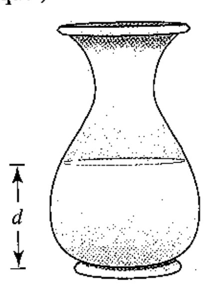


Figure for 58

58. ***Think About It*** Water is running into the vase shown in the figure at a constant rate.

(a) Graph the depth d of water in the vase as a function of time.

(b) Does the function have any extrema? Explain.

(c) Interpret the inflection points of the graph of d.

59. ***Conjecture*** Consider the function $f(x) = (x - 2)^n$.

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.

(b) Verify your conjecture in part (a).

60. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.

(b) Does $f''(x)$ exist at the inflection point? Explain.

In Exercises 61 and 62, find a, b, c, and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the indicated conditions.

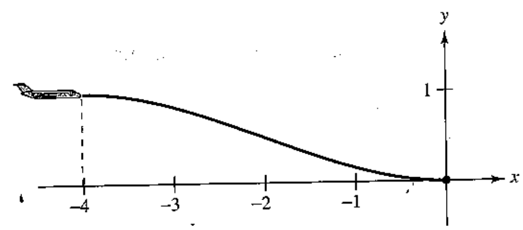
61. Relative maximum: $(3, 3)$
Relative minimum: $(5, 1)$
Inflection point: $(4, 2)$

62. Relative maximum: $(2, 4)$
Relative minimum: $(4, 2)$
Inflection point: $(3, 3)$

63. ***Aircraft Glide Path*** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).

(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.

(b) The function in part (a) models the glide path of the plane. When would the plane be descending at the most rapid rate?



FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

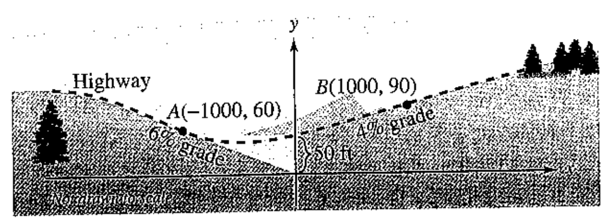
64. ***Highway Design*** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.

(a) Design a section of highway connecting the hillsides modeled by the function $f(x) = ax^3 + bx^2 + cx + d$ $(-1000 \leq x \leq 1000)$. At the points A and B, the slope of the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.

(c) Use a graphing utility to graph the derivative of the model.

(d) Determine the grade at the steepest part of the transitional section of the highway.



65. ***Beam Deflection*** The deflection D of a particular beam of length L is

$$D = 2x^4 - 5Lx^3 + 3L^2x^2$$

where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

66. ***Specific Gravity*** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \quad 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use a computer algebra system to find the coordinates of the maximum value of the function.

(b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(c) Estimate the specific gravity of water when $T = 20°$.

67. ***Average Cost*** A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

68. ***Inventory Cost*** The total cost C for ordering and storing x units is

$$C = 2x + \frac{300{,}000}{x}.$$

What order size will produce a minimum cost?

69. ***Sales Growth*** The annual sales S of a new product is given by

$$S = \frac{5000t^2}{8 + t^2}$$

where t is time in years. Find the time when sales are increasing at the greatest rate.

70. ***Modeling Data*** The average typing speed S of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? Combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 71–74, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	*Value of a*
71. $f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
72. $f(x) = 2(\sin x + \cos x)$	$a = 0$
73. $f(x) = \sqrt{1 - x}$	$a = 0$
74. $f(x) = \dfrac{\sqrt{x}}{x - 1}$	$a = 2$

75. Use a graphing utility to graph $y = x\sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.

76. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

77. Prove that every cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

78. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

True or False? **In Exercises 79–84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. The graph of every cubic polynomial has precisely one point of inflection.

80. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

81. The maximum value of $y = 3\sin x + 2\cos x$ is 5.

82. The maximum slope of the graph of $y = \sin(bx)$ is b.

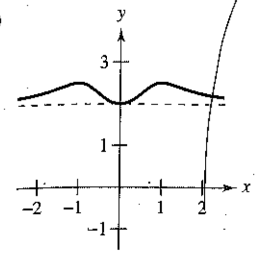
83. If $f'(c) > 0$, then f is concave upward at $x = c$.

84. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

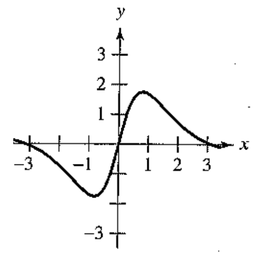
EXERCISES FOR SECTION 3.5

In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

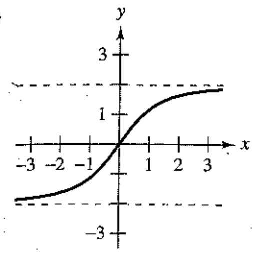
(a)



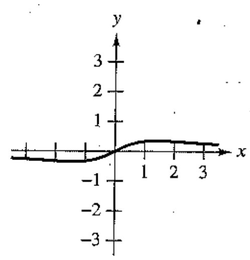
(b)



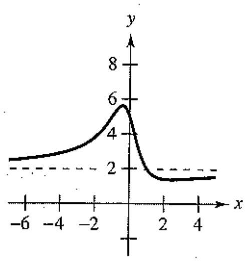
(c)



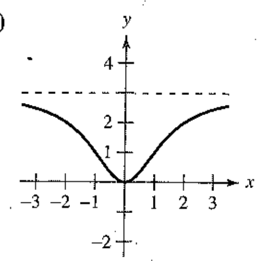
(d)



(e)



(f)



1. $f(x) = \dfrac{3x^2}{x^2+2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

3. $f(x) = \dfrac{x}{x^2+2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4+1}$

5. $f(x) = \dfrac{4\sin x}{x^2+1}$

6. $f(x) = \dfrac{2x^2-3x+5}{x^2+1}$

Numerical and Graphical Analysis **In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x+3}{2x-1}$

8. $f(x) = \dfrac{2x^2}{x+1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$

10. $f(x) = \dfrac{8x}{\sqrt{x^2-3}}$

11. $f(x) = 5 - \dfrac{1}{x^2+1}$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

In Exercises 13 and 14, find $\lim\limits_{x\to\infty} h(x)$, if possible.

13. $f(x) = 5x^3 - 3x^2 + 10$

(a) $h(x) = \dfrac{f(x)}{x^2}$ (b) $h(x) = \dfrac{f(x)}{x^3}$

(c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = 5x^2 - 3x + 7$

(a) $h(x) = \dfrac{f(x)}{x}$ (b) $h(x) = \dfrac{f(x)}{x^2}$

(c) $h(x) = \dfrac{f(x)}{x^3}$

In Exercises 15–18, find each of the limits, if possible.

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^2-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x-1}$

16. (a) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

17. (a) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^2-4}$

(b) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^{3/2}-4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x-4}$

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2+1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2}+1}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x}+1}$

In Exercises 19–32, find the limit.

19. $\lim\limits_{x\to\infty} \dfrac{2x-1}{3x+2}$

20. $\lim\limits_{x\to\infty} \dfrac{3x^3+2}{9x^3-2x^2+7}$

21. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1}$

22. $\lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right)$

23. $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x+3}$

24. $\lim\limits_{x\to-\infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right)$

25. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}}$

26. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

27. $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

28. $\lim\limits_{x\to-\infty} \dfrac{-3x+1}{\sqrt{x^2+x}}$

29. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

30. $\lim\limits_{x\to\infty} \dfrac{x-\cos x}{x}$

31. $\lim\limits_{x\to\infty} \dfrac{1}{2x+\sin x}$

32. $\lim\limits_{x\to\infty} \cos\dfrac{1}{x}$

In Exercises 33 and 34, use a graphing utility to graph the function and verify that it has two horizontal asymptotes.

33. $f(x) = \dfrac{|x|}{x+1}$

34. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

In Exercises 35 and 36, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)

35. $\lim_{x\to\infty} x \sin \frac{1}{x}$

36. $\lim_{x\to\infty} x \tan \frac{1}{x}$

In Exercises 37–40, find the limit. (*Hint:* Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

37. $\lim_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

38. $\lim_{x\to\infty} \left(2x - \sqrt{4x^2+1}\right)$

39. $\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

40. $\lim_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 41–44, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

41. $f(x) = x - \sqrt{x(x-1)}$

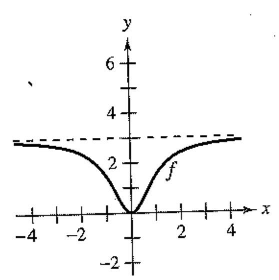
42. $f(x) = x^2 - x\sqrt{x(x-1)}$

43. $f(x) = x \sin \frac{1}{2x}$

44. $f(x) = \frac{x+1}{x\sqrt{x}}$

Getting at the Concept

45. The graph of a function f is shown below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) Sketch f'.

(b) Use the graphs to estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

46. Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$

$f'(x) > 0$ for $x > 2$

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$

47. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 46 and has *no* points of inflection? Explain.

Getting at the Concept *(continued)*

48. If f is a continuous function such that $\lim_{x\to\infty} f(x) = 5$, find, if possible, $\lim_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric to the y-axis.

(b) The graph of f is symmetric to the origin.

In Exercises 49–66, sketch the graph of the equation. Look for extrema, intercepts, symmetry, and asymptotes as necessary. Use a graphing utility to verify your result.

49. $y = \frac{2+x}{1-x}$

50. $y = \frac{x-3}{x-2}$

51. $y = \frac{x}{x^2-4}$

52. $y = \frac{2x}{9-x^2}$

53. $y = \frac{x^2}{x^2+9}$

54. $y = \frac{x^2}{x^2-9}$

55. $y = \frac{2x^2}{x^2-4}$

56. $y = \frac{2x^2}{x^2+4}$

57. $xy^2 = 4$

58. $x^2y = 4$

59. $y = \frac{2x}{1-x}$

60. $y = \frac{2x}{1-x^2}$

61. $y = 2 - \frac{3}{x^2}$

62. $y = 1 + \frac{1}{x}$

63. $y = 3 + \frac{2}{x}$

64. $y = 4\left(1 - \frac{1}{x^2}\right)$

65. $y = \frac{x^3}{\sqrt{x^2-4}}$

66. $y = \frac{x}{\sqrt{x^2-4}}$

In Exercises 67–76, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.

67. $f(x) = 5 - \frac{1}{x^2}$

68. $f(x) = \frac{x^2}{x^2-1}$

69. $f(x) = \frac{x}{x^2-4}$

70. $f(x) = \frac{1}{x^2-x-2}$

71. $f(x) = \frac{x-2}{x^2-4x+3}$

72. $f(x) = \frac{x+1}{x^2+x+1}$

73. $f(x) = \frac{3x}{\sqrt{4x^2+1}}$

74. $g(x) = \frac{2x}{\sqrt{3x^2+1}}$

75. $g(x) = \sin \frac{x}{x-2}, \quad 3 < x < \infty$

76. $f(x) = \frac{2 \sin 2x}{x}$

In Exercises 77 and 78, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)

77. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$

$g(x) = x + \dfrac{2}{x(x-3)}$

78. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

79. ***Average Cost*** A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

80. ***Engine Efficiency*** The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

81. A line with slope m passes through the point $(0, 4)$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

82. ***Modeling Data*** The table shows the world record times for running one mile, where t represents the year with $t = 0$ corresponding to 1900, and y is the time in minutes and seconds.

t	23	33	45	54
y	4:10.4	4:07.6	4:01.3	3:59.4

t	58	66	79	85	99
y	3:54.5	3:51.3	3:48.9	3:46.3	3:43.1

A model for the data is

$$y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$

where the seconds have been changed to a decimal part of a minute.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting time for running one mile? Explain.

83. ***Modeling Data*** A heat probe is attached to the heat exchanger of a heating system. The temperature T (degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is

$$T_2 = \frac{1451 + 86t}{58 + t}.$$

Use a graphing utility to graph the model.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim_{t\to\infty} T_2$.

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

84. ***Modeling Data*** The average typing speed S of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}, \quad t > 0.$

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

85. In your own words, state the guidelines for finding the limit of a rational function. Give examples.

86. Prove that if $p(x) = a_nx^n + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + \cdots + b_1x + b_0$ $(a_n \neq 0, b_m \neq 0)$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m. \end{cases}$$

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

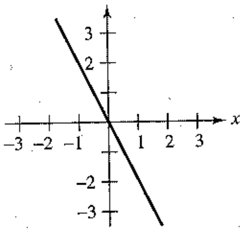
88. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.

EXERCISES FOR SECTION 3.6

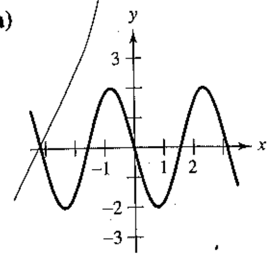
In Exercises 1–4, match the graph of f in the left column with that of its derivative in the right column.

Graph of f | *Graph of f'*

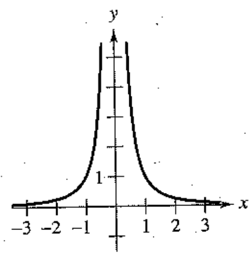
1.



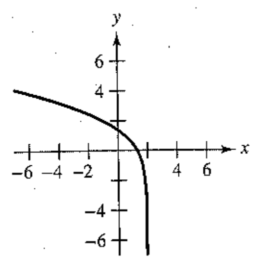
(a)



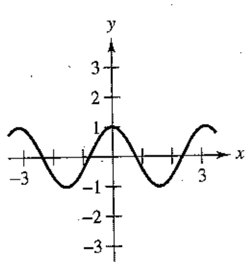
2.



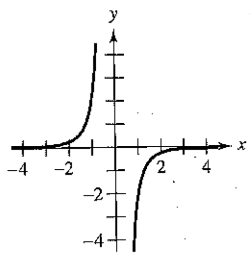
(b)



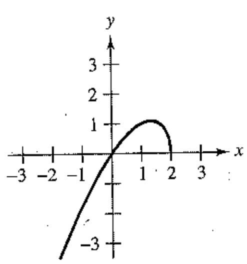
3.



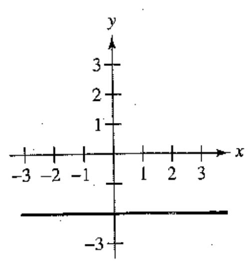
(c)



4.



(d)



5. ***Graphical Reasoning*** The graph of f is given in the figure.
   (a) For which values of x is $f'(x)$ zero? Positive? Negative?
   (b) For which values of x is $f''(x)$ zero? Positive? Negative?
   (c) On what interval is f' an increasing function?
   (d) For which value of x is $f'(x)$ minimum? For this value of x, how does the rate of change of f compare with the rate of change of f for other values of x? Explain.

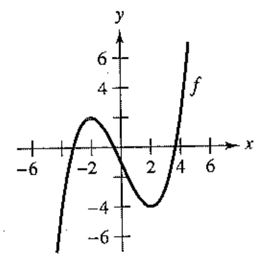


Figure for 5

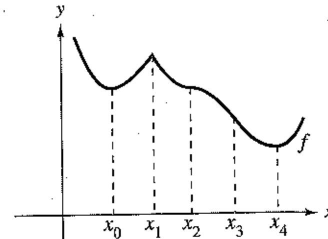


Figure for 6

6. ***Graphical Reasoning*** Identify the real numbers x_0, x_1, x_2, x_3, and x_4 in the figure such that each of the following is true.
   (a) $f'(x) = 0$ (b) $f''(x) = 0$
   (c) $f'(x)$ does not exist. (d) f has a relative maximum.
   (e) f has a point of inflection.

In Exercises 7–38, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

7. $y = \dfrac{x^2}{x^2 + 3}$
8. $y = \dfrac{x}{x^2 + 1}$
9. $y = \dfrac{1}{x - 2} - 3$
10. $y = \dfrac{x^2 + 1}{x^2 - 9}$
11. $y = \dfrac{2x}{x^2 - 1}$
12. $f(x) = \dfrac{x + 2}{x}$
13. $g(x) = x + \dfrac{4}{x^2 + 1}$
14. $f(x) = x + \dfrac{32}{x^2}$
15. $f(x) = \dfrac{x^2 + 1}{x}$
16. $f(x) = \dfrac{x^3}{x^2 - 4}$
17. $y = \dfrac{x^2 - 6x + 12}{x - 4}$
18. $y = \dfrac{2x^2 - 5x + 5}{x - 2}$
19. $y = x\sqrt{4 - x}$
20. $g(x) = x\sqrt{9 - x}$
21. $h(x) = x\sqrt{9 - x^2}$
22. $y = x\sqrt{16 - x^2}$
23. $y = 3x^{2/3} - 2x$
24. $y = 3(x - 1)^{2/3} - (x - 1)^2$
25. $y = x^3 - 3x^2 + 3$
26. $y = -\frac{1}{3}(x^3 - 3x + 2)$
27. $y = 2 - x - x^3$
28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$
29. $f(x) = 3x^3 - 9x + 1$
30. $f(x) = (x + 1)(x - 2)(x - 5)$
31. $y = 3x^4 + 4x^3$
32. $y = 3x^4 - 6x^2 + \frac{5}{3}$
33. $f(x) = x^4 - 4x^3 + 16x$
34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$
35. $y = x^5 - 5x$
36. $y = (x - 1)^5$
37. $y = |2x - 3|$
38. $y = |x^2 - 6x + 5|$

In Exercises 39–46, sketch a graph of the function over the indicated interval. Use a graphing utility to verify your graph.

Function	*Interval*
39. $y = \sin x - \frac{1}{18}\sin 3x$	$0 \le x \le 2\pi$
40. $y = \cos x - \frac{1}{2}\cos 2x$	$0 \le x \le 2\pi$
41. $y = 2x - \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
42. $y = 2(x - 2) + \cot x$	$0 < x < \pi$
43. $y = 2(\csc x + \sec x)$	$0 < x < \frac{\pi}{2}$
44. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2\tan\left(\frac{\pi x}{8}\right) - 1$	$-3 < x < 3$
45. $g(x) = x\tan x$	$-\frac{3\pi}{2} < x < \frac{3\pi}{2}$
46. $g(x) = x\cot x$	$-2\pi < x < 2\pi$

In Exercises 47–50, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.

47. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$

48. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

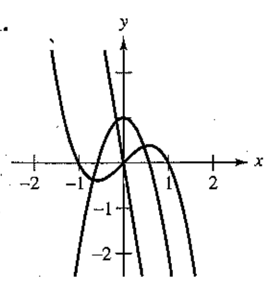
49. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$

50. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

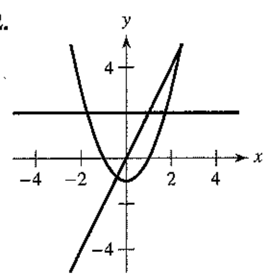
Getting at the Concept

In Exercises 51 and 52, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

51.

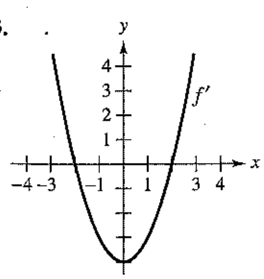


52.

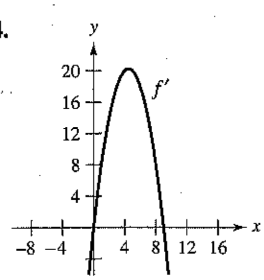


In Exercises 53–56, use the graph of f' to sketch a graph of f and the graph of f''. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

53.

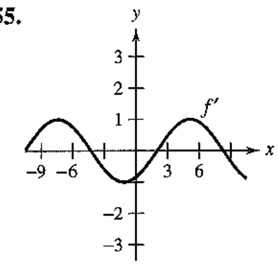


54.

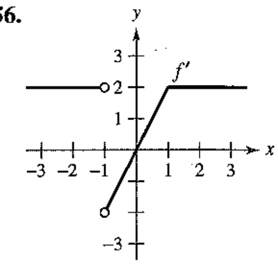


Getting at the Concept *(continued)*

55.



56.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

57. Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

58. Suppose $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

In Exercises 59 and 60, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

59. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$

60. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Writing **In Exercises 61 and 62, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

61. $h(x) = \dfrac{6 - 2x}{3 - x}$

62. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

Writing **In Exercises 63 and 64, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?**

63. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2}$

64. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5}$

65. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad 0 < x < 4.$$

(a) Use a computer algebra system to graph the function and use the graph to visually approximate the critical numbers.

(b) Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

66. ***Graphical Reasoning*** Consider the function

$$f(x) = \tan(\sin \pi x).$$

(a) Use a graphing utility to graph the function.

(b) Identify any symmetry of the graph.

(c) Is the function periodic? If so, what is the period?

(d) Identify any extrema on $(-1, 1)$.

(e) Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It **In Exercises 67–70, create a function whose graph has the indicated characteristics. (There is more than one correct answer.)**

67. Vertical asymptote: $x = 5$
Horizontal asymptote: $y = 0$

68. Vertical asymptote: $x = -3$
Horizontal asymptote: None

69. Vertical asymptote: $x = 5$
Slant asymptote: $y = 3x + 2$

70. Vertical asymptote: $x = 0$
Slant asymptote: $y = -x$

71. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{ax}{(x-b)^2}.$$

(a) Determine the effect on the graph of f if $b \neq 0$ and a is varied. Consider cases where a is positive and a is negative.

(b) Determine the effect on the graph of f if $a \neq 0$ and b is varied.

72. Consider the function

$$f(x) = \tfrac{1}{2}(ax)^2 - (ax), \quad a \neq 0.$$

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

(b) In the same viewing window, use a graphing utility to graph the function for four different values of a.

73. ***Investigation*** Consider the function

$$f(x) = \frac{3x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

(c) For which value of n will $y = 3$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

Table for 73(e)

74. ***Investigation*** Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as indicated in the figure. Verify each of the following.

(a) The x-intercept of the tangent line is $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$.

(b) The y-intercept of the tangent line is $(0, f(x_0) - x_0 f'(x_0))$.

(c) The x-intercept of the normal line is $(x_0 + f(x_0)f'(x_0), 0)$.

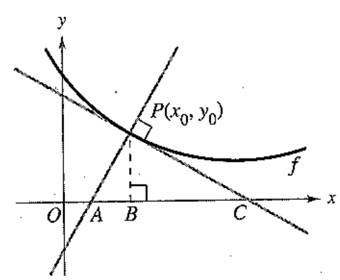
(d) The y-intercept of the normal line is $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$.

(e) $|BC| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC| = \left|\dfrac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$

(g) $|AB| = |f(x_0)f'(x_0)|$

(h) $|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$



75. ***Modeling Data*** The data in the table show the number N of bacteria in a culture at time t, where t is measured in days.

t	1	2	3	4
N	25	200	804	1756

t	5	6	7	8
N	2296	2434	2467	2473

A model for this data is given by

$$N = \frac{24{,}670 - 35{,}153t + 13{,}250t^2}{100 - 39t + 7t^2}, \quad 1 \le t \le 8.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the model to estimate the number of bacteria when $t = 10$.

(c) Approximate the day when the number of bacteria was greatest.

(d) Use a computer algebra system to determine the time when the rate of increase in the number of bacteria was greatest.

(e) Find $\lim_{t\to\infty} N(t)$.

Lab Series LAB 5

EXERCISES FOR SECTION 3.7

1. ***Numerical, Graphical, and Analytic Analysis*** Find two positive numbers whose sum is 110 and whose product is a maximum.
   (a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

   (b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the solution. (*Hint:* Use the *table* feature of the graphing utility.)
   (c) Write the product P as a function of x.
   (d) Use a graphing utility to graph the function in part (c) and estimate the solution from the graph.
   (e) Use calculus to find the critical number of the function in part (c). Then find the two numbers.

In Exercises 2–6, find two positive numbers that satisfy the given requirements.

2. The sum is S and the product is a maximum.
3. The product is 192 and the sum is a minimum.
4. The product is 192 and the sum of the first plus three times the second is a minimum.
5. The second number is the reciprocal of the first and the sum is a minimum.
6. The sum of the first and twice the second is 100 and the product is a maximum.

In Exercises 7 and 8, find the length and width of a rectangle that has the given perimeter and a maximum area.

7. Perimeter: 100 meters
8. Perimeter: P units

In Exercises 9 and 10, find the length and width of a rectangle that has the given area and a minimum perimeter.

9. Area: 64 square feet
10. Area: A square centimeters

In Exercises 11–14, find the point on the graph of the function that is closest to the given point.

Function	*Point*	*Function*	*Point*
11. $f(x) = \sqrt{x}$	$(4, 0)$	12. $f(x) = \sqrt{x - 8}$	$(2, 0)$
13. $f(x) = x^2$	$\left(2, \frac{1}{2}\right)$	14. $f(x) = (x + 1)^2$	$(5, 3)$

15. ***Chemical Reaction*** In an autocatalytic chemical reaction, the product formed is a catalyst for the reaction. If Q_0 is the amount of the original substance and x is the amount of catalyst formed, the rate of chemical reaction is

$$\frac{dQ}{dx} = kx(Q_0 - x).$$

For what value of x will the rate of chemical reaction be greatest?

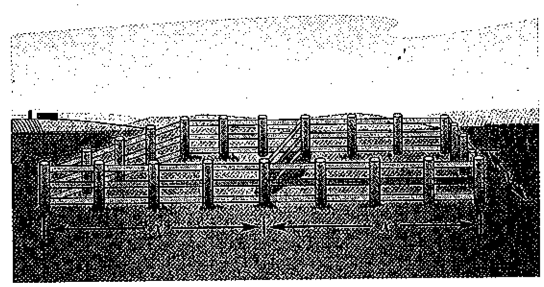
16. ***Traffic Control*** On a given day, the flow rate F (cars per hour) on a congested roadway is

$$F = \frac{v}{22 + 0.02v^2}$$

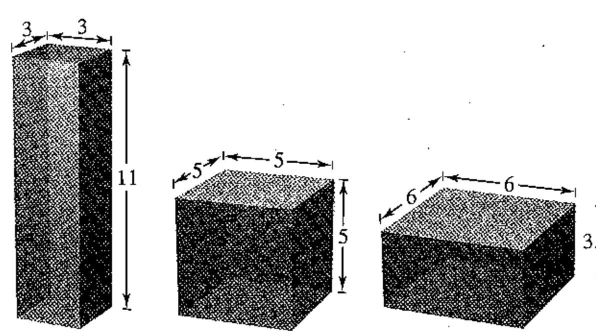
where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

17. ***Area*** A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

18. ***Area*** A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



19. ***Volume***
   (a) Verify that each of the rectangular solids shown in the figure has a surface area of 150 square inches.
   (b) Find the volume of each.
   (c) Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.



20. ***Numerical, Graphical, and Analytic Analysis*** An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).

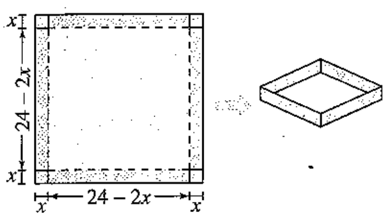
(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height	Length and Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

(b) Write the volume V as a function of x.

(c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

(d) Use a graphing utility to graph the function in part (b) and verify the maximum volume from the graph.



21. (a) Solve Exercise 20 given that the square piece of material is s meters on a side.

(b) If the dimensions of the square piece of material are doubled, how does the volume change?

22. ***Numerical, Graphical, and Analytic Analysis*** A physical fitness room consists of a rectangle with a semicircle on each end. A 200-meter running track runs around the outside of the room.

(a) Draw a figure to represent the problem. Let x and y represent the length and width of the rectangle.

(b) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum area of the rectangular region.

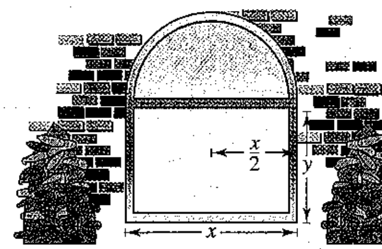
Length x	Width y	Area
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$

(c) Write the area A as a function of x.

(d) Use calculus to find the critical number of the function in part (c) and find the maximum value.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum area from the graph.

23. ***Area*** A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.



24. ***Area*** A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?

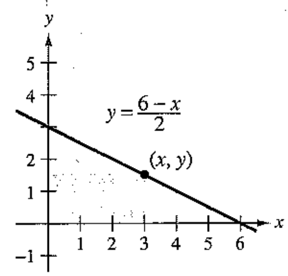


Figure for 24

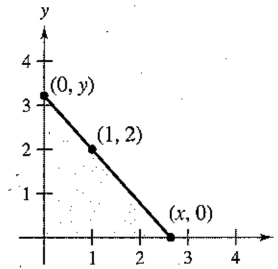


Figure for 25

25. ***Length*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1, 2)$ (see figure).

(a) Write the length L of the hypotenuse as a function of x.

(b) Use a graphing utility to graphically approximate x such that the length of the hypotenuse is a minimum.

(c) Find the vertices of the triangle such that its area is a minimum.

26. ***Area*** Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 4 (see figure).

(a) Solve by writing the area as a function of h.

(b) Solve by writing the area as a function of α.

(c) Identify the type of triangle of maximum area.

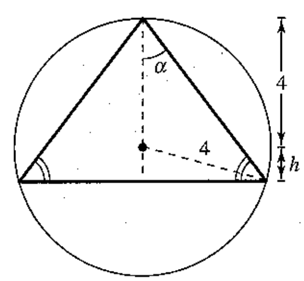


Figure for 26

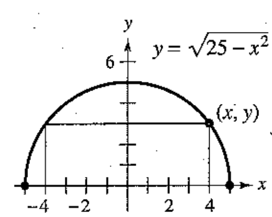


Figure for 27

27. ***Area*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

28. ***Area*** Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 27).

29. ***Area*** A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

30. ***Area*** A rectangular page is to contain 36 square inches of print. The margins on each side are to be $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

31. ***Numerical, Graphical, and Analytic Analysis*** A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Radius r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.1$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the surface area S as a function of r.

(d) Use a graphing utility to graph the function in part (c) and estimate the minimum surface area from the graph.

(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

32. ***Surface Area*** Use calculus to find the required dimensions for the cylinder in Exercise 31 if its volume is V_0 cubic units.

33. ***Volume*** A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)

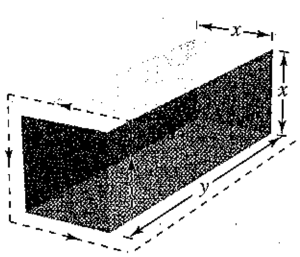


Figure for 33

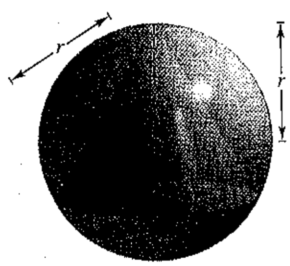


Figure for 35

34. ***Volume*** Rework Exercise 33 for a cylindrical package. (The cross section is circular.)

35. ***Volume*** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r (see figure).

36. ***Volume*** Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

Getting at the Concept

37. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.

38. A plastic shampoo bottle is a right circular cylinder. Because the surface area of the bottle does not change when it is squeezed, is it true that the volume remains the same? Explain.

39. ***Surface Area*** A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 12 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

40. ***Cost*** An industrial tank of the shape described in Exercise 39 must have a volume of 3000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

41. ***Area*** The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce a minimum total area.

42. ***Area*** Twenty feet of wire is to be used to form two figures. In each of the following cases, how much should be used for each figure so that the total enclosed area is maximum?

(a) Equilateral triangle and square

(b) Square and regular pentagon

(c) Regular pentagon and regular hexagon

(d) Regular hexagon and circle

What can you conclude from this pattern? {*Hint:* The area of a regular polygon with n sides of length x is $A = (n/4)[\cot(\pi/n)]x^2$.}

43. ***Beam Strength*** A wooden beam has a rectangular cross section of height h and width w (see figure). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 24 inches? (*Hint:* $S = kh^2w$, where k is the proportionality constant.)

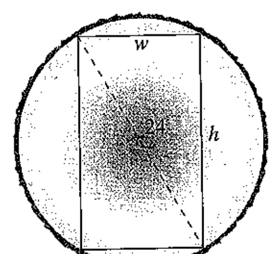


Figure for 43

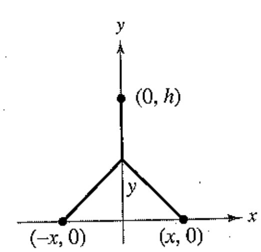


Figure for 44

44. ***Minimum Length*** Two factories are located at the coordinates $(-x, 0)$ and $(x, 0)$ with their power supply located at the point $(0, h)$ (see figure). Find y such that the total amount of power line from the power supply to the factories is a minimum.

45. ***Projectile Range*** The range R of a projectile fired with an initial velocity v_0 at an angle θ with the horizontal is

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

where g is the acceleration due to gravity. Find the angle θ such that the range is a maximum.

46. ***Conjecture*** Consider the functions $f(x) = \frac{1}{2}x^2$ and $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on the domain $[0, 4]$.

(a) Use a graphing utility to graph the functions on the specified domain.

(b) Write the vertical distance d between the functions as a function of x and use calculus to find the value of x for which d is maximum.

(c) Find the equations of the tangent lines to the graphs of f and g at the critical number found in part (b). Graph the tangent lines. What is the relationship between the lines?

(d) Make a conjecture about the relationship between tangent lines to the graphs of two functions at the value of x at which the vertical distance between the functions is greatest, and prove your conjecture.

47. ***Illumination*** A light source is located over the center of a circular table of diameter 4 feet (see figure). Find the height h of the light source such that the illumination I at the perimeter of the table is maximum if $I = k(\sin \alpha)/s^2$, where s is the slant height, α is the angle at which the light strikes the table, and k is a constant.

48. ***Illumination*** The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. What point on the line segment joining the two sources has the least illumination?

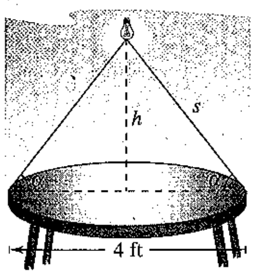


Figure for 47

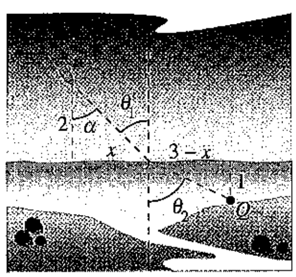


Figure for 49

49. ***Minimum Time*** A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). If he can row at 2 miles per hour and walk at 4 miles per hour, toward what point on the coast should he row in order to reach point Q in the least time?

50. ***Minimum Time*** Consider Exercise 49 if the point Q is on the shoreline rather than 1 mile inland.

(a) Write the travel time T as a function of α.

(b) Use the result of part (a) to find the minimum time to reach Q.

(c) Suppose the man can row at v_1 miles per hour and walk at v_2 miles per hour. Write the time T as a function of α. Show that the critical number of T depends only on v_1 and v_2 and not the distances. Explain how this result would be more beneficial to the man than the result of Exercise 49.

(d) Describe how to apply the result of part (c) to minimizing the cost of constructing a power transmission cable that costs c_1 dollars per mile under water and c_2 dollars per mile over land.

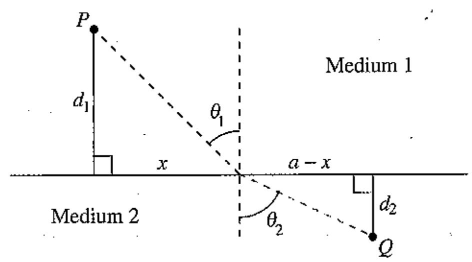
51. ***Minimum Time*** The conditions are the same as in Exercise 49 except that the man can row at v_1 miles per hour and walk at v_2 miles per hour. If θ_1 and θ_2 are the magnitudes of the angles, show that the man will reach point Q in the least time when

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

52. ***Minimum Time*** When light waves, traveling in a transparent medium, strike the surface of a second transparent medium, they change directions. This change of direction is called *refraction* and is defined by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure and v_1 and v_2 are the velocities of light in the two media. Show that this problem is equivalent to that of Exercise 51, and that light waves traveling from P to Q follow the path of minimum time.



53. Sketch the graph of $f(x) = 2 - 2 \sin x$ on the interval $[0, \pi/2]$.

(a) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

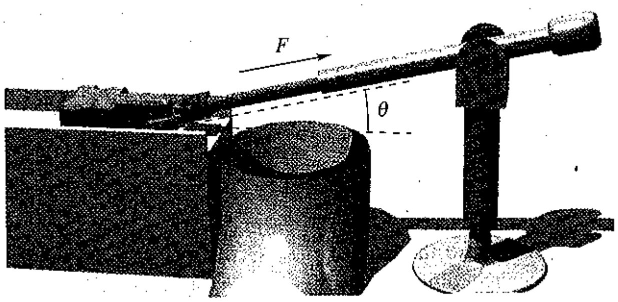
(b) Express the distance d from the origin to a point on the graph of f as a function of x. Use your graphing utility to graph d and find the minimum distance.

(c) Use calculus and the root finding capabilities of a graphing utility to find the value of x that minimizes the function d on the interval $[0, \pi/2]$. What is the minimum distance?

(Submitted by Tim Chapell, Penn Valley Community College, Kansas City, MO.)

54. ***Minimum Cost*** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. If laying pipe in the ocean is twice as expensive as on land, what path should the pipe follow in order to minimize the cost?

55. ***Minimum Force*** A component is designed to slide a block of steel with weight W across a table and into a chute (see figure.) The motion of the block is resisted by a frictional force proportional to its apparent weight. (Let k be the constant of proportionality.) Find the minimum force F needed to slide the block, and find the corresponding value of θ. (*Hint:* $F \cos \theta$ is the force in the direction of motion, and $F \sin \theta$ is the amount of force tending to lift the block. Therefore, the apparent weight of the block is $W - F \sin \theta$.)



56. ***Volume*** A sector with central angle θ is cut from a circle of radius 12 inches (see figure), and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

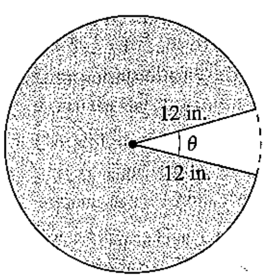


Figure for 56

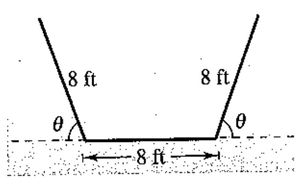


Figure for 57

57. ***Numerical, Graphical, and Analytic Analysis*** The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long (see figure). Determine the angle of elevation θ of the sides such that the area of the cross section is a maximum by completing the following.

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10°$	$8 \sin 10°$	≈ 22.1
8	$8 + 16 \cos 20°$	$8 \sin 20°$	≈ 42.5

(b) Use a graphing utility to generate additional rows of the table and estimate the maximum cross-sectional area. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the cross-sectional area A as a function of θ.

(d) Use calculus to find the critical number of the function in part (c) and find the angle that will yield the maximum cross-sectional area.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum cross-sectional area.

58. ***Maximum Profit*** Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Furthermore, the bank can reinvest this money at 12%. Find the interest rate the bank should pay to maximize profit. (Use the simple interest formula.)

59. ***Minimum Cost*** The ordering and transportation cost C of the components used in manufacturing a certain product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \qquad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the order size that minimizes the cost. (*Hint:* Use the *root* feature of a graphing utility.)

60. ***Diminishing Returns*** The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is

$$P = -\tfrac{1}{10}s^3 + 6s^2 + 400.$$

(a) Find the amount of money the company should spend on advertising in order to yield a maximum profit.

(b) The *point of diminishing returns* is the point at which the rate of growth of the profit function begins to decline. Find the point of diminishing returns.

Minimum Distance **In Exercises 61–63, consider a fuel distribution center located at the origin of the rectangular coordinate system (units in miles; see figures). The center supplies three factories with coordinates (4, 1), (5, 6), and (10, 3). A trunk line will run from the distribution center along the line $y = mx$, and feeder lines will run to the three factories. The objective is to find m such that the lengths of the feeder lines are minimized.**

61. Minimize the sum of the squares of the lengths of vertical feeder lines given by

$$S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2.$$

Find the equation for the trunk line by this method and then determine the sum of the lengths of the feeder lines.

62. Minimize the sum of the absolute values of the lengths of vertical feeder lines given by

$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|.$$

Find the equation for the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_2 and approximate the required critical number.)

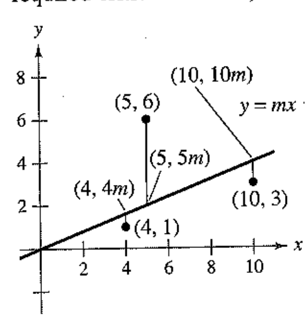
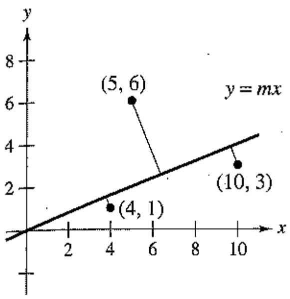


Figure for 61 and 62

63. Minimize the sum of the perpendicular distances (see Exercises 83–88 in Section P.2) from the trunk line to the factories given by

$$S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}.$$

Find the equation for the trunk line by this method and then determine the sum of the lengths of the feeder lines. (*Hint:* Use a graphing utility to graph the function S_3 and approximate the required critical number.)

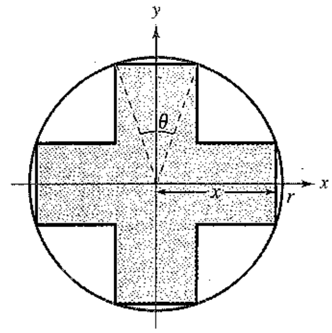


64. ***Area*** Consider a symmetric cross inscribed in a circle of radius r (see figure).

(a) Write the area A of the cross as a function of x and find the value of x that maximizes the area.

(b) Write the area A of the cross as a function of θ and find the value of θ that maximizes the area.

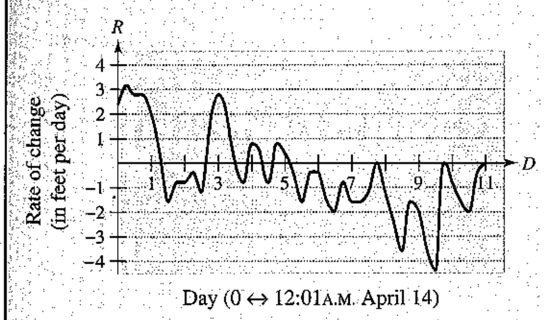
(c) Show that the critical numbers of parts (a) and (b) yield the same maximum area. What is that area?



SECTION PROJECT CONNECTICUT RIVER

Whenever the Connecticut River reaches a level of 105 feet above sea level, two Northampton, Massachusetts flood control station operators begin a round-the-clock river watch. Every two hours, they check the height of the river, using a scale marked off in tenths of a foot, and record the data in a log book. In the spring of 1996, the flood watch lasted from April 4, when the river reached 105 feet and was rising at 0.2 foot per hour, until April 25, when the level subsided again to 105 feet. Between those dates, their log shows that the river rose and fell several times, at one point coming close to the 115-foot mark. If the river had reached 115 feet, the city would have closed down Mount Tom Road (Route 5, south of Northampton).

The graph below shows the *rate of change* of the level of the river during one portion of the flood watch. Use the graph to answer the following questions.



(a) On what date was the river rising most rapidly? How do you know?

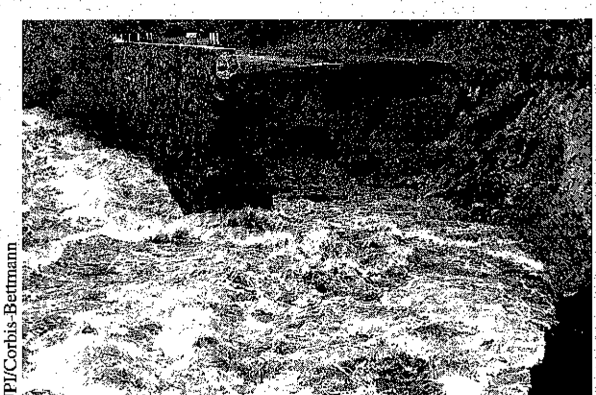
(b) On what date was the river falling most rapidly? How do you know?

(c) There were two dates in a row on which the river rose, then fell, then rose again during the course of the day. On which days did this occur, and how do you know?

(d) At one minute past midnight, April 14, the river level was 111.0 feet. Estimate its height 24 hours later and 48 hours later. Explain how you made your estimates.

(e) The river crested at 114.4 feet. On what date do you think this occurred?

(Submitted by Mary Murphy, Smith College, Northampton, MA)



UPI/Corbis-Bettmann

EXERCISES FOR SECTION 3.8

In Exercises 1–4, complete two iterations of Newton's Method for the function using the indicated initial guess.

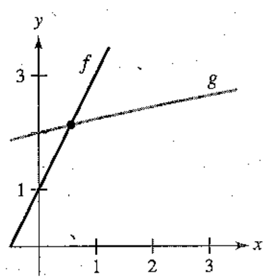
1. $f(x) = x^2 - 3, \quad x_1 = 1.7$
2. $f(x) = 2x^2 - 3, \quad x_1 = 1$
3. $f(x) = \sin x, \quad x_1 = 3$
4. $f(x) = \tan x, \quad x_1 = 0.1$

In Exercises 5–14, approximate the zero(s) of the function. Use Newton's Method and continue the process until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

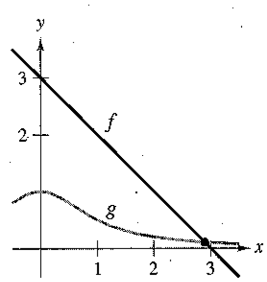
5. $f(x) = x^3 + x - 1$
6. $f(x) = x^5 + x - 1$
7. $f(x) = 3\sqrt{x-1} - x$
8. $f(x) = x - 2\sqrt{x+1}$
9. $f(x) = x^3 + 3$
10. $f(x) = 1 - 2x^3$
11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$
12. $f(x) = \frac{1}{2}x^4 - 3x - 3$
13. $f(x) = x + \sin(x + 1)$
14. $f(x) = x^3 - \cos x$

In Exercises 15–18, apply Newton's Method to approximate the x-value of the indicated point(s) of intersection of the two graphs. Continue the process until two successive approximations differ by less than 0.001. [*Hint:* Let $h(x) = f(x) - g(x)$.]

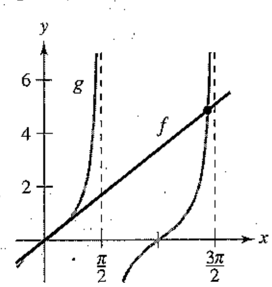
15. $f(x) = 2x + 1$
$g(x) = \sqrt{x + 4}$



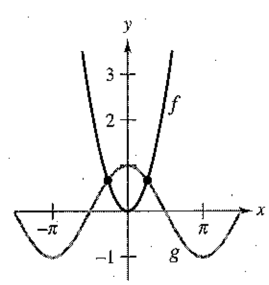
16. $f(x) = 3 - x$
$g(x) = 1/(x^2 + 1)$



17. $f(x) = x$
$g(x) = \tan x$



18. $f(x) = x^2$
$g(x) = \cos x$



In Exercises 19 and 20, use Newton's Method to obtain a general rule for approximating the required radical.

19. $x = \sqrt{a}$ [*Hint:* Consider $f(x) = x^2 - a$.]
20. $x = \sqrt[n]{a}$ [*Hint:* Consider $f(x) = x^n - a$.]

In Exercises 21–24, use the results of Exercises 19 and 20 to approximate the indicated radical to three decimal places.

21. $\sqrt{7}$
22. $\sqrt{5}$
23. $\sqrt[4]{6}$
24. $\sqrt[3]{15}$

In Exercises 25 and 26, approximate π to three decimal places using Newton's Method and the given function.

25. $f(x) = 1 + \cos x$
26. $f(x) = \tan x$

In Exercises 27–30, apply Newton's Method using the indicated initial guess, and explain why the method fails.

27. $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$
28. $y = 4x^3 - 12x^2 + 12x - 3, \quad x_1 = \frac{3}{2}$

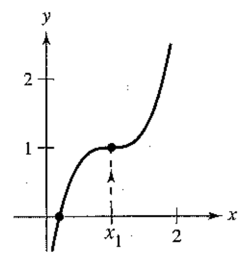


Figure for 27

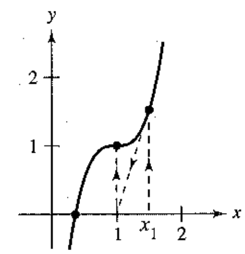


Figure for 28

29. $f(x) = -x^3 + 6x^2 - 10x + 6, \quad x_1 = 2$
30. $f(x) = 2\sin x + \cos 2x, \quad x_1 = \frac{3\pi}{2}$

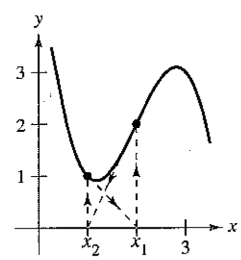


Figure for 29

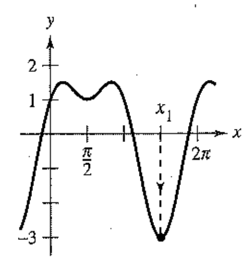


Figure for 30

Getting at the Concept

31. In your own words and using a sketch, describe Newton's Method for approximating the zeros of a function.
32. Under what conditions will Newton's Method fail?

Fixed Point **In Exercises 33 and 34, approximate the fixed point of the function to two decimal places. [A *fixed point* x_0 of a function f is a value of x such that $f(x_0) = x_0$.]**

33. $f(x) = \cos x$
34. $f(x) = \cot x, \quad 0 < x < \pi$

35. *Writing* Consider the function $f(x) = x^3 - 3x^2 + 3$.
(a) Use a graphing utility to obtain the graph of f.
(b) Use Newton's Method with $x_1 = 1$ as an initial guess.
(c) Repeat part (b) using $x_1 = \frac{1}{4}$ as an initial guess and observe that the result is different.
(d) To understand why the results in parts (b) and (c) are different, sketch the tangent lines to the graph of f at the points $(1, f(1))$ and $\left(\frac{1}{4}, f\left(\frac{1}{4}\right)\right)$. Find the x-intercept of each tangent line and compare the intercepts with the first iteration of Newton's Method using the respective initial guesses.
(e) Write a short paragraph summarizing how Newton's Method works. Use the results of this exercise to describe why it is important to select the initial guess carefully.

36. *Writing* Repeat the steps in Exercise 35 for the function $f(x) = \sin x$ with initial guesses of $x_1 = 1.8$ and $x_1 = 3$.

37. Use Newton's Method to show that the equation

$$x_{n+1} = x_n(2 - ax_n)$$

can be used to approximate $1/a$ if x_1 is an initial guess of the reciprocal of a. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

38. Use the result of Exercise 37 to approximate the indicated reciprocal to three decimal places.
(a) $\frac{1}{3}$ (b) $\frac{1}{11}$

In Exercises 39 and 40, approximate the critical number of f on the interval $(0, \pi)$. Sketch the graph of f, labeling any extrema.

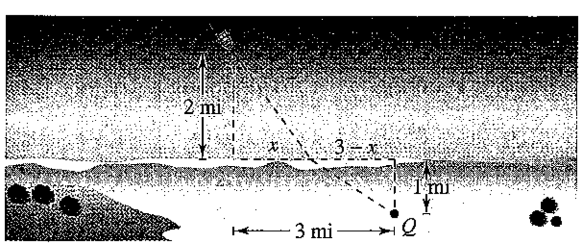
39. $f(x) = x \cos x$
40. $f(x) = x \sin x$

In Exercises 41–44, we review some typical problems from the previous sections of this chapter. In each case, use Newton's Method to approximate the solution.

41. ***Minimum Distance*** Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

42. ***Minimum Distance*** Find the point on the graph of $f(x) = x^2$ that is closest to the point $(4, -3)$.

43. ***Minimum Time*** You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point Q, which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach Q in the least time?



44. ***Medicine*** The concentration C of a certain chemical in the bloodstream t hours after injection into muscle tissue is given by $C = (3t^2 + t)/(50 + t^3)$. When is the concentration greatest?

45. ***Advertising Costs*** A company that produces portable cassette players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where P is the profit in dollars and x is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of two advertising amounts that yield a profit P of \$2,500,000.

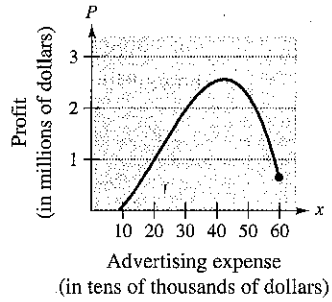


Figure for 45

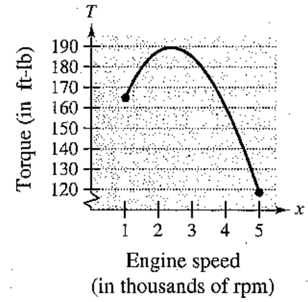


Figure for 46

46. ***Engine Power*** The torque produced by a compact automobile engine is approximated by the model

$$T = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, \quad 1 \le x \le 5$$

where T is the torque in foot-pounds and x is the engine speed in thousands of revolutions per minute (see figure). Approximate the two engine speeds that yield a torque T of 170 foot-pounds.

True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. The zeros of $f(x) = p(x)/q(x)$ coincide with the zeros of $p(x)$.

48. If the coefficients of a polynomial function are all positive, then the polynomial has no positive zeros.

49. If $f(x)$ is a cubic polynomial such that $f'(x)$ is never zero, then any initial guess will force Newton's Method to converge to the zero of f.

50. The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

In Exercises 51 and 52, *write a computer program or use a spreadsheet* to find the zeros of a function using Newton's Method. Approximate the zeros of the function accurate to three decimal places. The output should be a table with the following headings.

$$n, \quad x_n, \quad f(x_n), \quad f'(x_n), \quad \frac{f(x_n)}{f'(x_n)}, \quad x_n - \frac{f(x_n)}{f'(x_n)}$$

51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

52. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

EXERCISES FOR SECTION 3.9

In Exercises 1–6, find the equation of the tangent line T to the graph of f at the indicated point. Use this linear approximation to complete the table.

x	1.9	1.99	2	2.01	2.1
$f(x)$					
$T(x)$					

Function	*Point*
1. $f(x) = x^2$	$(2, 4)$
2. $f(x) = \dfrac{6}{x^2}$	$\left(2, \dfrac{3}{2}\right)$
3. $f(x) = x^5$	$(2, 32)$
4. $f(x) = \sqrt{x}$	$(2, \sqrt{2})$
5. $f(x) = \sin x$	$(2, \sin 2)$
6. $f(x) = \csc x$	$(2, \csc 2)$

In Exercises 7–10, use the information to evaluate and compare Δy and dy.

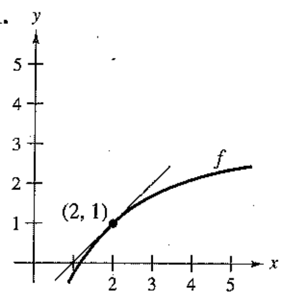
7. $y = \frac{1}{2}x^3$ $\quad x = 2 \quad \Delta x = dx = 0.1$
8. $y = 1 - 2x^2$ $\quad x = 0 \quad \Delta x = dx = -0.1$
9. $y = x^4 + 1$ $\quad x = -1 \quad \Delta x = dx = 0.01$
10. $y = 2x + 1$ $\quad x = 2 \quad \Delta x = dx = 0.01$

In Exercises 11–20, find the differential dy of the given function.

11. $y = 3x^2 - 4$
12. $y = 3x^{2/3}$
13. $y = \dfrac{x + 1}{2x - 1}$
14. $y = \sqrt{9 - x^2}$
15. $y = x\sqrt{1 - x^2}$
16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
17. $y = 2x - \cot^2 x$
18. $y = x \sin x$
19. $y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$
20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

In Exercises 21–24, use differentials and the graph of f to approximate (a) $f(1.9)$ and (b) $f(2.04)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

21.



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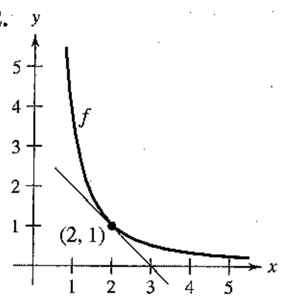

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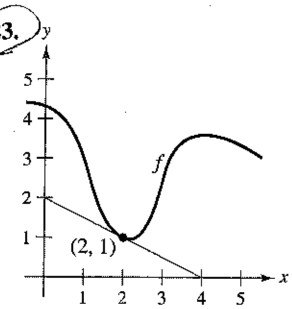

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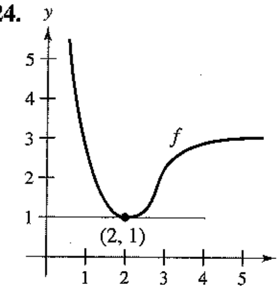
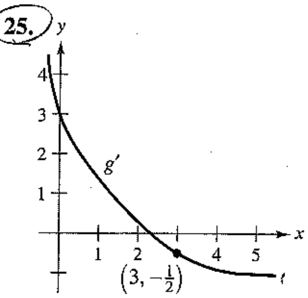


In Exercises 25–28, use differentials and the graph of g' to approximate (a) $g(2.93)$ and (b) $g(3.1)$ given that $g(3) = 8$.

25.



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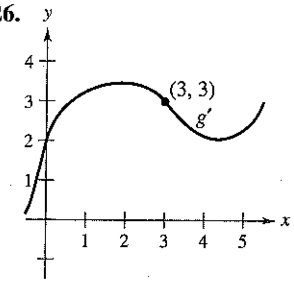

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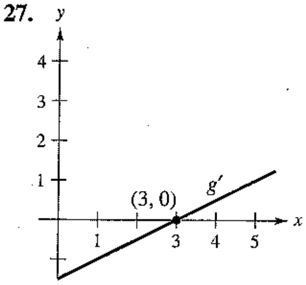

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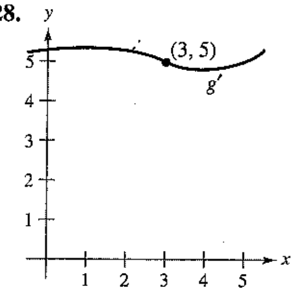

29. ***Area*** The measurement of the side of a square is found to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

30. ***Area*** The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters. The possible error in each measurement is 0.25 centimeter. Use differentials to approximate the possible propagated error in computing the area of the triangle.

31. ***Area*** The measurement of the radius of the end of a log is found to be 14 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the possible propagated error in computing the area of the end of the log.

32. ***Volume and Surface Area*** The measurement of the edge of a cube is found to be 12 inches, with a possible error of 0.03 inch. Use differentials to approximate the maximum possible propagated error in computing
    (a) the volume of the cube.
    (b) the surface area of the cube.

33. ***Area*** The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

(a) Approximate the percent error in computing the area of the square.

(b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

34. ***Circumference*** The measurement of the circumference of a circle is found to be 56 centimeters. The possible error in measuring the circumference is 1.2 centimeters.

(a) Approximate the percent error in computing the area of the circle.

(b) Estimate the maximum allowable percent error in measuring the circumference if the error in computing the area cannot exceed 3%.

35. ***Volume and Surface Area*** The radius of a sphere is measured to be 6 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere, (b) the surface area of the sphere, and (c) the relative errors in parts (a) and (b).

36. ***Profit*** The profit P for a company is given by

$$P = (500x - x^2) - \left(\tfrac{1}{2}x^2 - 77x + 3000\right).$$

Approximate the change and percent change in profit as production changes from $x = 115$ to $x = 120$ units.

In Exercises 37 and 38, the thickness of the shell is 0.2 centimeter. Use differentials to approximate the volume of the shell.

37. A cylindrical shell with height 40 centimeters and radius 5 centimeters

38. A spherical shell of radius 100 centimeters

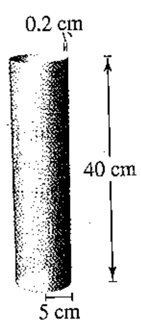

Figure for 37

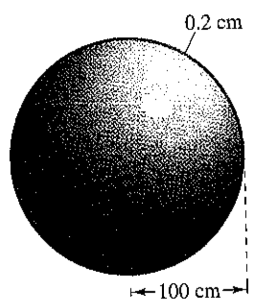

Figure for 38

39. ***Pendulum*** The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. Suppose that the pendulum has been subjected to an increase in temperature such that the length has increased by $\frac{1}{2}\%$.

(a) Find the approximate percent change in the period.

(b) Using the result in part (a), find the approximate error in this pendulum clock in one day.

40. ***Ohm's Law*** A current of I amperes passes through a resistor of R ohms. **Ohm's Law** states that the voltage E applied to the resistor is

$$E = IR.$$

If the voltage is constant, show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I.

41. ***Triangle Measurements*** The measurement of one side of a right triangle is found to be 9.5 inches, and the angle opposite that side is 26°45′ with a possible error of 15′.

(a) Approximate the percent error in computing the length of the hypotenuse.

(b) Estimate the maximum allowable percent error in measuring the angle if the error in computing the length of the hypotenuse cannot exceed 2%.

42. ***Area*** Approximate the percent error in computing the area of the triangle in Exercise 41.

43. ***Projectile Motion*** The range R of a projectile is

$$R = \frac{v_0^2}{32}(\sin 2\theta)$$

where v_0 is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2200$ feet per second and θ is changed from 10° to 11°, use differentials to approximate the change in the range.

44. ***Surveying*** A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

In Exercises 45–48, use differentials to approximate the value of the expression. Compare your answer with that of a calculator.

45. $\sqrt{99.4}$ **46.** $\sqrt[3]{26}$ **47.** $\sqrt[4]{624}$ **48.** $(2.99)^3$

Writing **In Exercises 49 and 50, give a short explanation of why the approximation is valid.**

49. $\sqrt{4.02} \approx 2 + \frac{1}{4}(0.02)$ **50.** $\tan 0.05 \approx 0 + 1(0.05)$

Getting at the Concept

51. Describe the change in accuracy of dy as an approximation for Δy when Δx is decreased.

52. When using differentials, what is meant by the terms propagated error, relative error, and percent error?

True or False? **In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

53. If $y = x + c$, then $dy = dx$.

54. If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

55. If y is differentiable, then $\lim_{\Delta x \to 0} (\Delta y - dy) = 0$.

56. If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.

REVIEW EXERCISES FOR CHAPTER 3

3.1

1. Give the definition of a critical number, and graph a function f showing the different types of critical numbers.

2. Consider the odd function f that is continuous, differentiable, and has the functional values shown in the table.

x	-5	-4	-1	0	2	3	6
$f(x)$	1	3	2	0	-1	-4	0

(a) Determine $f(4)$.

(b) Determine $f(-3)$.

(c) Plot the points and make a possible sketch of the graph of f on the interval $[-6, 6]$. What is the smallest number of critical points in the interval? Explain.

(d) Does there exist at least one real number c in the interval $(-6, 6)$ where $f'(c) = -1$? Explain.

(e) Is it possible that $\lim_{x \to 0} f(x)$ does not exist? Explain.

(f) Is it necessary that $f'(x)$ exists at $x = 2$? Explain.

In Exercises 3 and 4, find the absolute extrema of the function on the closed interval. Use a graphing utility to graph the function over the indicated interval to confirm your results.

3. $g(x) = 2x + 5 \cos x, \quad [0, 2\pi]$

4. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, \quad [0, 2]$

3.2 **In Exercises 5 and 6, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$.**

5. $f(x) = (x - 2)(x + 3)^2, \quad [-3, 2]$

6. $f(x) = |x - 2| - 2, \quad [0, 4]$

7. Consider the function $f(x) = 3 - |x - 4|$.

(a) Graph the function and verify that $f(1) = f(7)$.

(b) Note that $f'(x)$ is not equal to zero for any x in $[1, 7]$. Explain why this does not contradict Rolle's Theorem.

8. Can the Mean Value Theorem be applied to the function $f(x) = 1/x^2$ on the interval $[-2, 1]$? Explain.

In Exercises 9–12, find the point(s) guaranteed by the Mean Value Theorem for the closed interval $[a, b]$.

9. $f(x) = x^{2/3}, \quad [1, 8]$

10. $f(x) = \dfrac{1}{x}, \quad [1, 4]$

11. $f(x) = x - \cos x, \quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

12. $f(x) = \sqrt{x} - 2x, \quad [0, 4]$

13. For the function $f(x) = Ax^2 + Bx + C$, determine the value of c guaranteed by the Mean Value Theorem on the interval $[x_1, x_2]$.

14. Demonstrate the result of Exercise 13 for $f(x) = 2x^2 - 3x + 1$ on the interval $[0, 4]$.

3.3 **In Exercises 15–18, find the critical numbers (if any) and the open intervals on which the function is increasing or decreasing.**

15. $f(x) = (x - 1)^2(x - 3)$

16. $g(x) = (x + 1)^3$

17. $h(x) = \sqrt{x}(x - 3), \quad x > 0$

18. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

In Exercises 19 and 20, use the First Derivative Test to find any relative extrema of the function. Use a graphing utility to verify your results.

19. $h(t) = \dfrac{1}{4}t^4 - 8t$

20. $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \quad [0, 4]$

21. ***Harmonic Motion*** The height of an object attached to a spring is given by the harmonic equation

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in inches and t is measured in seconds.

(a) Calculate the height and velocity of the object when $t = \pi/8$ second.

(b) Show that the maximum displacement of the object is $\frac{5}{12}$ inch.

(c) Find the period P of y. Also, find the frequency f (number of oscillations per second) if $f = 1/P$.

22. ***Writing*** The general equation giving the height of an oscillating object attached to a spring is

$$y = A \sin \sqrt{\frac{k}{m}}\,t + B \cos \sqrt{\frac{k}{m}}\,t$$

where k is the spring constant and m is the mass of the object.

(a) Show that the maximum displacement of the object is $\sqrt{A^2 + B^2}$.

(b) Show that the object oscillates with a frequency of

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

3.4 **In Exercises 23 and 24, determine the points of inflection of the function.**

23. $f(x) = x + \cos x, \quad [0, 2\pi]$

24. $f(x) = (x + 2)^2(x - 4)$

In Exercises 25 and 26, use the Second Derivative Test to find all relative extrema.

25. $g(x) = 2x^2(1 - x^2)$

26. $h(t) = t - 4\sqrt{t + 1}$

Think About It **In Exercises 27 and 28, sketch the graph of a function f having the indicated characteristics.**

27. $f(0) = f(6) = 0$
$f'(3) = f'(5) = 0$
$f'(x) > 0$ if $x < 3$
$f'(x) > 0$ if $3 < x < 5$
$f'(x) < 0$ if $x > 5$
$f''(x) < 0$ if $x < 3$ and $x > 4$
$f''(x) > 0,\ 3 < x < 4$

28. $f(0) = 4,\ f(6) = 0$
$f'(x) < 0$ if $x < 2$ and $x > 4$
$f'(2)$ does not exist.
$f'(4) = 0$
$f'(x) > 0$ if $2 < x < 4$
$f''(x) < 0,\ x \neq 2$

29. ***Writing*** A newspaper headline states that "The rate of growth of the national deficit is decreasing." What does this mean? What does it imply about the graph of the deficit as a function of time?

30. ***Inventory Cost*** The cost of inventory depends on the ordering and storage costs according to the inventory model

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r.$$

Determine the order size that will minimize the cost, assuming that sales occur at a constant rate, Q is the number of units sold per year, r is the cost of storing one unit for 1 year, s is the cost of placing an order, and x is the number of units per order.

31. ***Modeling Data*** Outlays for national defense D (in billions of dollars) for selected years from 1970 through 1999 are shown in the table, where t is time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. Office of Management and Budget)*

t	0	5	10	15	20
D	90.4	103.1	155.1	279.0	328.3

t	25	26	27	28	29
D	309.9	302.7	309.8	310.3	320.2

(a) Use the regression capabilities of a graphing utility to fit a model of the form $D = at^4 + bt^3 + ct^2 + dt + e$ to the data.
(b) Use a graphing utility to plot the data and graph the model.
(c) For the years shown in the table, when does the model indicate that the outlay for national defense is at a maximum? When is it at a minimum?
(d) For the years shown in the table, when does the model indicate that the outlay for national defense is increasing at the greatest rate?

32. ***Modeling Data*** The manager of a store recorded the annual sales S (in thousands of dollars) of a product over a period of 7 years, as shown in the table, where t is the time in years, with $t = 1$ corresponding to 1991.

t	1	2	3	4	5	6	7
S	5.4	6.9	11.5	15.5	19.0	22.0	23.6

(a) Use the regression capabilities of a graphing utility to find a model of the form $S = at^3 + bt^2 + ct + d$ for the data.
(b) Use a graphing utility to plot the data and graph the model.
(c) Use calculus to find the time t when sales were increasing at the greatest rate.
(d) Do you think the model would be accurate for predicting future sales? Explain.

3.5 **In Exercises 33–36, find the limit.**

33. $\lim_{x\to\infty} \dfrac{2x^2}{3x^2+5}$ **34.** $\lim_{x\to\infty} \dfrac{2x}{3x^2+5}$

35. $\lim_{x\to\infty} \dfrac{5\cos x}{x}$ **36.** $\lim_{x\to\infty} \dfrac{3x}{\sqrt{x^2+4}}$

In Exercises 37–40, find any vertical and horizontal asymptotes of the graph of the function. Use a graphing utility to verify your results.

37. $h(x) = \dfrac{2x+3}{x-4}$ **38.** $g(x) = \dfrac{5x^2}{x^2+2}$

39. $f(x) = \dfrac{3}{x} - 2$ **40.** $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

In Exercises 41–44, use a graphing utility to graph the function. Use the graph to approximate any relative extrema or asymptotes.

41. $f(x) = x^3 + \dfrac{243}{x}$ **42.** $f(x) = |x^3 - 3x^2 + 2x|$

43. $f(x) = \dfrac{x-1}{1+3x^2}$ **44.** $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

3.6 **In Exercises 45–62, analyze and sketch the graph of the function.**

45. $f(x) = 4x - x^2$ **46.** $f(x) = 4x^3 - x^4$

47. $f(x) = x\sqrt{16 - x^2}$ **48.** $f(x) = (x^2 - 4)^2$

49. $f(x) = (x-1)^3(x-3)^2$ **50.** $f(x) = (x-3)(x+2)^3$

51. $f(x) = x^{1/3}(x+3)^{2/3}$ **52.** $f(x) = (x-2)^{1/3}(x+1)^{2/3}$

53. $f(x) = \dfrac{x+1}{x-1}$

54. $f(x) = \dfrac{2x}{1+x^2}$

55. $f(x) = \dfrac{4}{1 + x^2}$

56. $f(x) = \dfrac{x^2}{1 + x^4}$

57. $f(x) = x^3 + x + \dfrac{4}{x}$

58. $f(x) = x^2 + \dfrac{1}{x}$

59. $f(x) = |x^2 - 9|$

60. $f(x) = |x - 1| + |x - 3|$

61. $f(x) = x + \cos x, \quad 0 \le x \le 2\pi$

62. $f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x), \quad -1 \le x \le 1$

63. Find the maximum and minimum points on the graph of

$$x^2 + 4y^2 - 2x - 16y + 13 = 0$$

(a) without using calculus.

(b) using calculus.

64. Consider the function $f(x) = x^n$ for positive integer values of n.

(a) For what values of n does the function have a relative minimum at the origin?

(b) For what values of n does the function have a point of inflection at the origin?

3.7

65. ***Minimum Distance*** At noon, ship A is 100 kilometers due east of ship B. Ship A is sailing west at 12 kilometers per hour, and ship B is sailing south at 10 kilometers per hour. At what time will the ships be nearest to each other, and what will this distance be?

66. ***Maximum Area*** Find the dimensions of the rectangle of maximum area, with sides parallel to the coordinate axes, that can be inscribed in the ellipse given by

$$\frac{x^2}{144} + \frac{y^2}{16} = 1.$$

67. ***Minimum Length*** A right triangle in the first quadrant has the coordinate axes as sides, and the hypotenuse passes through the point $(1, 8)$. Find the vertices of the triangle such that the length of the hypotenuse is minimum.

68. ***Minimum Length*** The wall of a building is to be braced by a beam that must pass over a parallel fence 5 feet high and 4 feet from the building. Find the length of the shortest beam that can be used.

69. ***Maximum Area*** Three sides of a trapezoid have the same length s. Of all such possible trapezoids, show that the one of maximum area has a fourth side of length $2s$.

70. ***Maximum Area*** Show that the greatest area of any rectangle inscribed in a triangle is one half that of the triangle.

71. ***Minimum Distance*** Find the length of the longest pipe that can be carried level around a right-angle corner at the intersection of two corridors of widths 4 feet and 6 feet. (Do not use trigonometry.)

72. ***Minimum Distance*** Rework Exercise 71, given corridors of widths a meters and b meters.

73. ***Minimum Distance*** A hallway of width 6 feet meets a hallway of width 9 feet at right angles. Find the length of the longest pipe that can be carried level around this corner. [*Hint:* If L is the length of the pipe, show that

$$L = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

where θ is the angle between the pipe and the wall of the narrower hallway.]

74. ***Minimum Distance*** Rework Exercise 73, given that one hallway is of width a meters and the other is of width b meters. Show that the result is the same as in Exercise 72.

Minimum Cost **In Exercises 75 and 76, find the speed v, in miles per hour, that will minimize costs on a 110-mile delivery trip. The cost per hour for fuel is C dollars, and the driver is paid W dollars per hour. (Assume there are no costs other than wages and fuel.)**

75. Fuel cost: $C = \dfrac{v^2}{600}$

Driver: $W = \$5$

76. Fuel cost: $C = \dfrac{v^2}{500}$

Driver: $W = \$7.50$

3.8 **In Exercises 77 and 78, use Newton's Method to approximate any real zeros of the function accurate to three decimal places. Use the root-finding capabilities of a graphing utility to verify your results.**

77. $f(x) = x^3 - 3x - 1$

78. $f(x) = x^3 + 2x + 1$

In Exercises 79 and 80, use Newton's Method to approximate, to three decimal places, the x-value of the points of intersection of the equations. Use a graphing utility to verify your results.

79. $y = x^4$
$y = x + 3$

80. $y = \sin \pi x$
$y = 1 - x$

3.9 **In Exercises 81 and 82, find the differential dy.**

81. $y = x(1 - \cos x)$

82. $y = \sqrt{36 - x^2}$

83. ***Surface Area and Volume*** The diameter of a sphere is measured to be 18 centimeters, with a maximum possible error of 0.05 centimeter. Use differentials to approximate the possible propagated error and percent error in calculating the surface area and the volume of the sphere.

84. ***Demand Function*** A company finds that the demand for its commodity is $p = 75 - \frac{1}{4}x$. If x changes from 7 to 8, find and compare the values of Δp and dp.