TECHNOLOGY Some graphing utilities, such as *Derive, Maple, Mathcad, Mathematica,* and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where Δx is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x\to c}[f(x) - f(c)] &= \lim_{x\to c}\left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right]\\ &= \left[\lim_{x\to c}(x - c)\right]\left[\lim_{x\to c}\frac{f(x) - f(c)}{x - c}\right]\\ &= (0)[f'(c)]\\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. So, f is continuous at $x = c$.

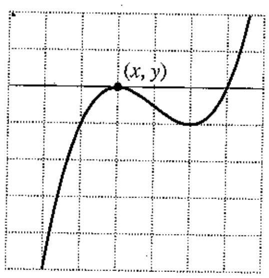
You can summarize the relationship between continuity and differentiability as follows.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

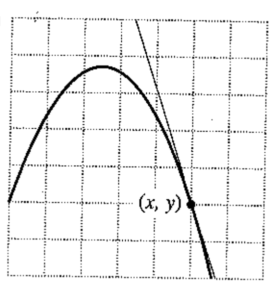
EXERCISES FOR SECTION 2.1

In Exercises 1 and 2, estimate the slope of the graph at the point (x, y).

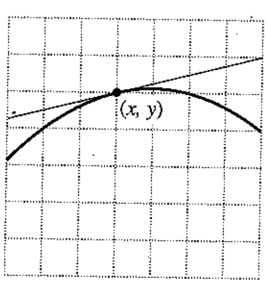
1. (a)



(b)



2. (a)



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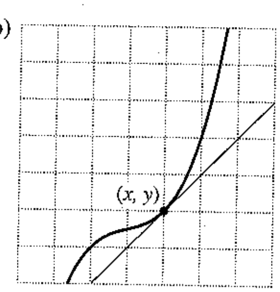
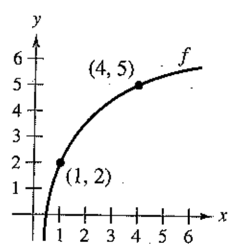


In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol (< or >) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1}$ ☐ $\dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1}$ ☐ $f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the specified point.

5. $f(x) = 3 - 2x, \quad (-1, 5)$
6. $g(x) = \frac{3}{2}x + 1, \quad (-2, -2)$
7. $g(x) = x^2 - 4, \quad (1, -3)$
8. $g(x) = 5 - x^2, \quad (2, 1)$
9. $f(t) = 3t - t^2, \quad (0, 0)$
10. $h(t) = t^2 + 3, \quad (-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11. $f(x) = 3$
12. $g(x) = -5$
13. $f(x) = -5x$
14. $f(x) = 3x + 2$
15. $h(s) = 3 + \frac{2}{3}s$
16. $f(x) = 9 - \frac{1}{2}x$
17. $f(x) = 2x^2 + x - 1$
18. $f(x) = 1 - x^2$
19. $f(x) = x^3 - 12x$
20. $f(x) = x^3 + x^2$
21. $f(x) = \dfrac{1}{x - 1}$
22. $f(x) = \dfrac{1}{x^2}$
23. $f(x) = \sqrt{x + 1}$
24. $f(x) = \dfrac{4}{\sqrt{x}}$

In Exercises 25–32, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

25. $f(x) = x^2 + 1, \quad (2, 5)$
26. $f(x) = x^2 + 2x + 1, \quad (-3, 4)$
27. $f(x) = x^3, \quad (2, 8)$
28. $f(x) = x^3 + 1, \quad (1, 2)$
29. $f(x) = \sqrt{x}, \quad (1, 1)$
30. $f(x) = \sqrt{x - 1}, \quad (5, 2)$
31. $f(x) = x + \dfrac{4}{x}, \quad (4, 5)$
32. $f(x) = \dfrac{1}{x + 1}, \quad (0, 1)$

In Exercises 33–36, find an equation of the line that is tangent to the graph of f *and* parallel to the given line.

Function	*Line*
33. $f(x) = x^3$	$3x - y + 1 = 0$
34. $f(x) = x^3 + 2$	$3x - y - 4 = 0$
35. $f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
36. $f(x) = \dfrac{1}{\sqrt{x - 1}}$	$x + 2y + 7 = 0$

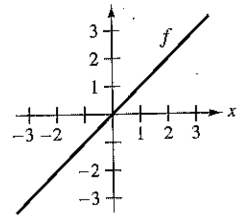
37. The tangent line to the graph of $y = g(x)$ at the point $(5, 2)$ passes through the point $(9, 0)$. Find $g(5)$ and $g'(5)$.

38. The tangent line to the graph of $y = h(x)$ at the point $(-1, 4)$ passes through the point $(3, 6)$. Find $h(-1)$ and $h'(-1)$.

Getting at the Concept

In Exercises 39–42, the graph of f is given. Select the graph of f'.

39.



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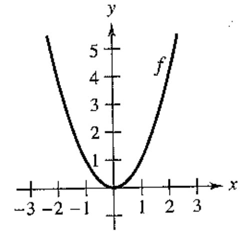

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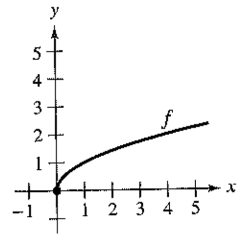

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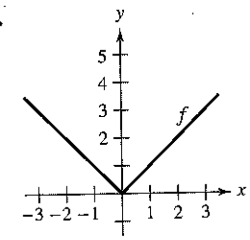
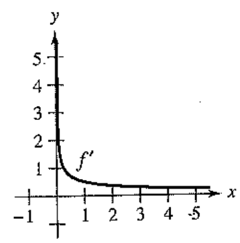


(a)



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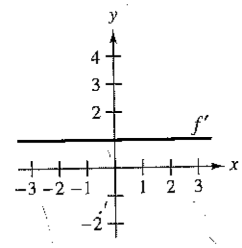
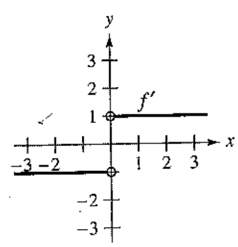
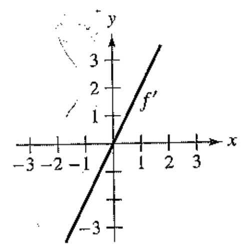


(c)



(d)



43. Sketch a graph of a function whose derivative is always negative.

44. Sketch a graph of a function whose derivative is always positive.

45. Assume that $f'(c) = 3$. Find $f'(-c)$ if (a) f is an odd function and if (b) f is an even function.

46. Determine whether the limit yields the derivative of a differentiable function f. Explain.

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + 2\Delta x) - f(x)}{2\Delta x}$

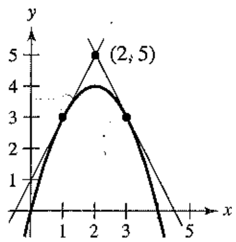
(b) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + 2) - f(x)}{\Delta x}$

(c) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$

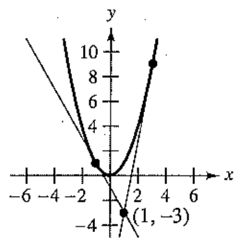
(d) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

In Exercises 47 and 48, find equations of the two tangent lines to the graph of f that pass through the indicated point.

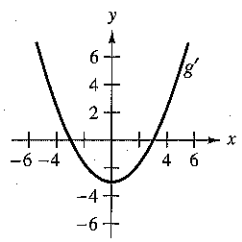
47. $f(x) = 4x - x^2$



48. $f(x) = x^2$



49. *Graphical Reasoning* The figure shows the graph of g'.



(a) $g'(0) =$

(b) $g'(3) =$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

50. *Graphical Reasoning* Use a graphing utility to graph each function and its tangent lines when $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slope of a tangent line to the graph of a function is always distinct for different values of x.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

Graphical, Numerical, and Analytic Analysis **In Exercises 51 and 52, use a graphing utility to graph f on the interval $[-2, 2]$. Complete the table by graphically estimating the slopes of the graph at the indicated points. Then evaluate the slopes analytically and compare your results with those obtained graphically.**

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

51. $f(x) = \frac{1}{4}x^3$ **52.** $f(x) = \frac{1}{2}x^2$

Graphical Reasoning **In Exercises 53 and 54, use a graphing utility to graph the functions f and g in the same viewing window where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

53. $f(x) = 2x - x^2$ **54.** $f(x) = 3\sqrt{x}$

In Exercises 55 and 56, evaluate $f(2)$ and $f(2.1)$ and use the results to approximate $f'(2)$.

55. $f(x) = x(4 - x)$ **56.** $f(x) = \frac{1}{4}x^3$

Graphical Reasoning **In Exercises 57 and 58, use a graphing utility to graph the function and its derivative in the same viewing window. Label the graphs and describe the relationship between them.**

57. $f(x) = \dfrac{1}{\sqrt{x}}$ **58.** $f(x) = \dfrac{x^3}{4} - 3x$

Writing **In Exercises 59 and 60, consider the functions f and $S_{\Delta x}$ where**

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2).$$

(a) Use a graphing utility to graph f and $S_{\Delta x}$ in the same viewing window for $\Delta x = 1, 0.5,$ and 0.1.

(b) Give a written description of the graphs of S for the different values of Δx in part (a).

59. $f(x) = 4 - (x - 3)^2$ **60.** $f(x) = x + \dfrac{1}{x}$

In Exercises 61–70, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

61. $f(x) = x^2 - 1,\ c = 2$ **62.** $g(x) = x(x - 1),\ c = 1$

63. $f(x) = x^3 + 2x^2 + 1,\ c = -2$

64. $f(x) = x^3 + 2x,\ c = 1$ **65.** $g(x) = \sqrt{|x|},\ c = 0$

66. $f(x) = 1/x,\ c = 3$ **67.** $f(x) = (x - 6)^{2/3},\ c = 6$

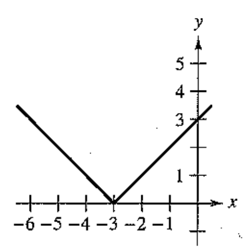
68. $g(x) = (x + 3)^{1/3},\ c = -3$

69. $h(x) = |x + 5|,\ c = -5$

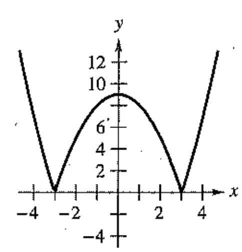
70. $f(x) = |x - 4|,\ c = 4$

In Exercises 71–80, describe the x-values at which f is differentiable.

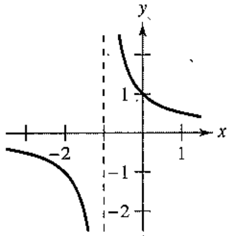
71. $f(x) = |x + 3|$



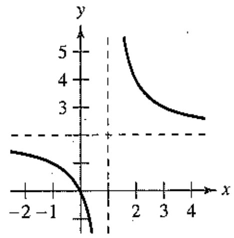
72. $f(x) = |x^2 - 9|$



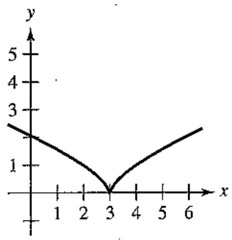
73. $f(x) = \dfrac{1}{x+1}$



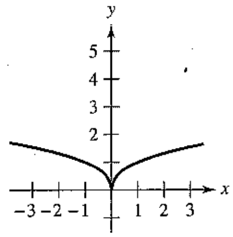
74. $f(x) = \dfrac{2x}{x-1}$



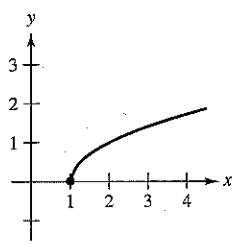
75. $f(x) = (x-3)^{2/3}$



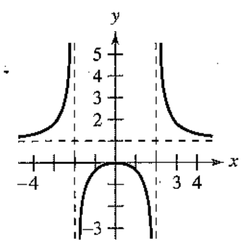
76. $f(x) = x^{2/5}$



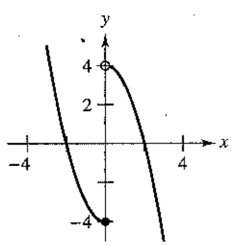
77. $f(x) = \sqrt{x-1}$



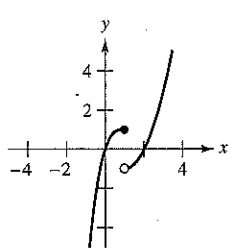
78. $f(x) = \dfrac{x^2}{x^2-4}$



79. $f(x) = \begin{cases} 4 - x^2, & x > 0 \\ x^2 - 4, & x \le 0 \end{cases}$



80. $f(x) = \begin{cases} x^2 - 2x, & x > 1 \\ x^3 - 3x^2 + 3x, & x \le 1 \end{cases}$



In Exercises 81–84, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

81. $f(x) = |x - 1|$

82. $f(x) = \sqrt{1 - x^2}$

83. $f(x) = \begin{cases} (x-1)^3, & x \le 1 \\ (x-1)^2, & x > 1 \end{cases}$

84. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 85 and 86, determine whether the function is differentiable at $x = 2$.

85. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

86. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

87. ***Graphical Reasoning*** A line with slope m passes through the point $(0, 4)$ and has the equation $y = mx + 4$.
   (a) Write the distance d between the line and the point $(3, 1)$ as a function of m.
   (b) Use a graphing utility to graph the function d in part (a). Based on the graph, is the function differentiable at every value of m? If not, where is it not differentiable?

88. ***Conjecture*** Consider the functions $f(x) = x^2$ and $g(x) = x^3$.
   (a) Graph f and f' on the same set of axes.
   (b) Graph g and g' on the same set of axes.
   (c) Identify any pattern between the functions f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.
   (d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. The slope of the tangent line to the differentiable function f at the point $(2, f(2))$ is
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

90. If a function is continuous at a point, then it is differentiable at that point.

91. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

92. If a function is differentiable at a point, then it is continuous at that point.

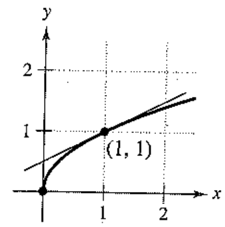
93. Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.
Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

94. ***Writing*** Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

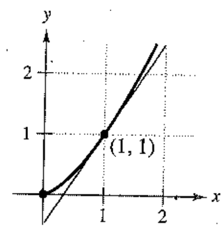
EXERCISES FOR SECTION 2.2

In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point (1, 1). Verify your answer analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

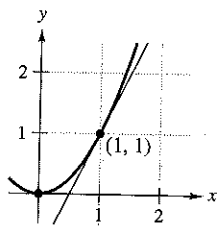
1. (a) $y = x^{1/2}$



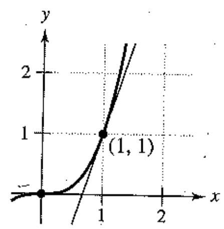
(b) $y = x^{3/2}$



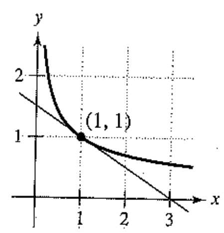
(c) $y = x^2$



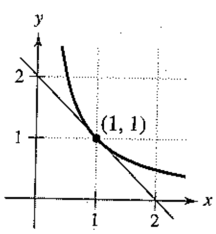
(d) $y = x^3$



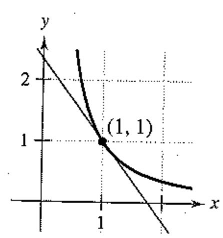
2. (a) $y = x^{-1/2}$



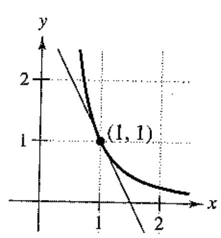
(b) $y = x^{-1}$



(c) $y = x^{-3/2}$



(d) $y = x^{-2}$



In Exercises 3–24, find the derivative of the function.

3. $y = 8$

4. $f(x) = -2$

5. $y = x^6$

6. $y = x^8$

7. $y = \dfrac{1}{x^7}$

8. $y = \dfrac{1}{x^8}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[4]{x}$

11. $f(x) = x + 1$

12. $g(x) = 3x - 1$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 + 2t - 3$

15. $g(x) = x^2 + 4x^3$

16. $y = 8 - x^3$

17. $s(t) = t^3 - 2t + 4$

18. $f(x) = 2x^3 - x^2 + 3x$

19. $y = \dfrac{\pi}{2}\sin\theta - \cos\theta$

20. $g(t) = \pi\cos t$

21. $y = x^2 - \frac{1}{2}\cos x$

22. $y = 5 + \sin x$

23. $y = \dfrac{1}{x} - 3\sin x$

24. $y = \dfrac{5}{(2x)^3} + 2\cos x$

In Exercises 25–30, complete the table, using Example 6 as a model.

Original Function	*Rewrite*	*Differentiate*	*Simplify*
25. $y = \dfrac{5}{2x^2}$			
26. $y = \dfrac{2}{3x^2}$			
27. $y = \dfrac{3}{(2x)^3}$			
28. $y = \dfrac{\pi}{(3x)^2}$			
29. $y = \dfrac{\sqrt{x}}{x}$			
30. $y = \dfrac{4}{x^{-3}}$			

In Exercises 31–38, find the slope of the graph of the function at the indicated point. Use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
31. $f(x) = \dfrac{3}{x^2}$	$(1, 3)$
32. $f(t) = 3 - \dfrac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34. $y = 3x^3 - 6$	$(2, 18)$
35. $y = (2x + 1)^2$	$(0, 1)$
36. $f(x) = 3(5 - x)^2$	$(5, 0)$
37. $f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38. $g(t) = 2 + 3\cos t$	$(\pi, -1)$

In Exercises 39–52, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$

40. $f(x) = x^2 - 3x - 3x^{-2}$

41. $g(t) = t^2 - \dfrac{4}{t^3}$

42. $f(x) = x + \dfrac{1}{x^2}$

43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$

44. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$

45. $y = x(x^2 + 1)$

46. $y = 3x(6x - 5x^2)$

47. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$

49. $h(s) = s^{4/5} - s^{2/3}$

50. $f(t) = t^{2/3} - t^{1/3} + 4$

51. $f(x) = 6\sqrt{x} + 5\cos x$

52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
53. $y = x^4 - 3x^2 + 2$	$(1, 0)$
54. $y = x^3 + x$	$(-1, -2)$
55. $f(x) = \dfrac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
56. $y = (x^2 + 2x)(x + 1)$	$(1, 6)$

In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

57. $y = x^4 - 8x^2 + 2$

58. $y = x^3 + x$

59. $y = \dfrac{1}{x^2}$

60. $y = x^2 + 1$

61. $y = x + \sin x, \quad 0 \le x < 2\pi$

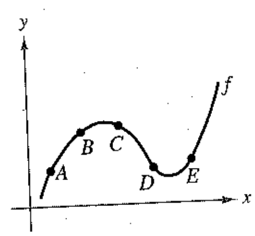
62. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

In Exercises 63–66, find k such that the line is tangent to the graph of the function.

Function	*Line*
63. $f(x) = x^2 - kx$	$y = 4x - 9$
64. $f(x) = k - x^2$	$y = -4x + 7$
65. $f(x) = \dfrac{k}{x}$	$y = -\dfrac{3}{4}x + 3$
66. $f(x) = k\sqrt{x}$	$y = x + 4$

Getting at the Concept

67. Use the graph of f to answer each question. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

68. Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

Getting at the Concept *(continued)*

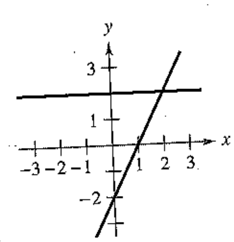
In Exercises 69 and 70, the relationship between f and g is given. Give the relationship between f' and g'.

69. $g(x) = f(x) + 6$

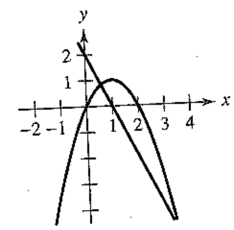
70. $g(x) = -5f(x)$

In Exercises 71 and 72, the graphs of a function f and its derivative f' are shown on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

71.



72.



73. Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

74. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their point of intersection.

In Exercises 75 and 76, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

75. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

76. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

77. ***Linear Approximation*** Use a graphing utility (in square mode) to zoom in on the graph of $f(x) = 4 - \frac{1}{2}x^2$ to approximate $f'(1)$. Use the derivative to find $f'(1)$.

78. ***Linear Approximation*** Use a graphing utility (in square mode) to zoom in on the graph of $f(x) = 4\sqrt{x} + 1$ to approximate $f'(4)$. Use the derivative to find $f'(4)$.

79. ***Linear Approximation*** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to obtain the graph of f. Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point "near" $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T on the same set of coordinate axes. Note that T is a "good" approximation of f when x is "close to" 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					

80. ***Linear Approximation*** Repeat Exercise 79 for the function $f(x) = x^3$ where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 79.

True or False? **In Exercises 81–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

81. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

82. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

83. If $y = \pi^2$, then $dy/dx = 2\pi$.

84. If $y = x/\pi$, then $dy/dx = 1/\pi$.

85. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

86. If $f(x) = 1/x^n$, then $f'(x) = 1/(nx^{n-1})$.

In Exercises 87–90, find the average rate of change of the function over the indicated interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

Function	*Interval*
87. $f(t) = 2t + 7$	$[1, 2]$
88. $f(t) = t^2 - 3$	$[2, 2.1]$
89. $f(x) = \dfrac{-1}{x}$	$[1, 2]$
90. $f(x) = \sin x$	$\left[0, \dfrac{\pi}{6}\right]$

Vertical Motion **In Exercises 91 and 92, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

91. A silver dollar is dropped from the top of a building that is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the coin at impact.

92. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

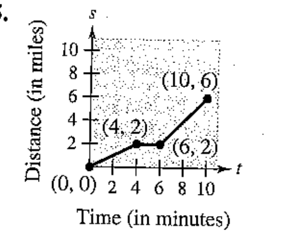
Vertical Motion **In Exercises 93 and 94, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

93. A projectile is shot upward from the surface of earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

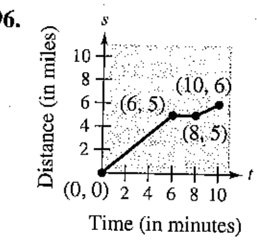
94. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

Think About It **In Exercises 95 and 96, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

95.

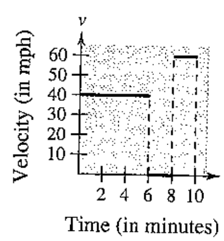


96.

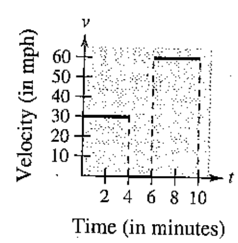


Think About It **In Exercises 97 and 98, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute drive to work. Make a sketch of the corresponding position function.**

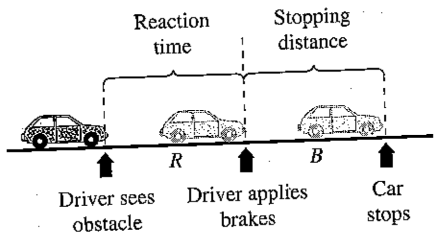
97.



98.



99. *Modeling Data* The stopping distance of an automobile traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



v	20	40	60	80	100
R	3.3	6.7	10.0	13.3	16.7
B	2.3	8.9	20.2	35.9	56.7

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking time.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing window.

(e) Find the derivative of T and the rate of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

100. *Velocity* Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

101. *Area* The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 4$ meters.

102. *Volume* The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$ centimeters.

103. *Inventory Management* The annual inventory cost C for a certain manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

104. *Fuel Cost* A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$1.25 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
$\frac{dC}{dx}$							

Who would benefit more from a 1-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon or the driver of a car that gets 35 miles per gallon? Explain.

105. *Writing* The number of gallons N of regular unleaded gasoline sold by a gasoline station at a price of p dollars per gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(1.479)$.

(b) Is $f'(1.479)$ usually positive or negative? Explain.

106. *Newton's Law of Cooling* This law states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature T and the temperature T_a of the surrounding medium. Write an equation for this law.

107. Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

108. Let (a, b) be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

109. Find the tangent line(s) to the curve $y = x^3 - 9x$ through the point $(1, -9)$.

110. Find the equation(s) of the tangent line(s) to the parabola $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

111. Find a and b such that

$$f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$$

is differentiable everywhere.

112. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

113. Prove that $\dfrac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in *Math Horizons*. To view this article, go to the website *www.matharticles.com*.

EXERCISES FOR SECTION 2.3

In Exercises 1–6, use the Product Rule to differentiate the function.

1. $g(x) = (x^2 + 1)(x^2 - 2x)$
2. $f(x) = (6x + 5)(x^3 - 2)$
3. $h(t) = \sqrt[3]{t}(t^2 + 4)$
4. $g(s) = \sqrt{s}(4 - s^2)$
5. $f(x) = x^3 \cos x$
6. $g(x) = \sqrt{x} \sin x$

In Exercises 7–12, use the Quotient Rule to differentiate the function.

7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $g(t) = \dfrac{t^2 + 2}{2t - 7}$
9. $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1}$
10. $h(s) = \dfrac{s}{\sqrt{s} - 1}$
11. $g(x) = \dfrac{\sin x}{x^2}$
12. $f(t) = \dfrac{\cos t}{t^3}$

In Exercises 13–18, find $f'(x)$ and $f'(c)$.

Function	*Value of c*
13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$	$c = 0$
14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$	$c = 1$
15. $f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
16. $f(x) = \dfrac{x + 1}{x - 1}$	$c = 2$
17. $f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
18. $f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

In Exercises 19–24, complete the table without using the Quotient Rule (see Example 6).

Function	*Rewrite*	*Differentiate*	*Simplify*
19. $y = \dfrac{x^2 + 2x}{3}$			
20. $y = \dfrac{5x^2 - 3}{4}$			
21. $y = \dfrac{7}{3x^3}$			
22. $y = \dfrac{4}{5x^2}$			
23. $y = \dfrac{4x^{3/2}}{x}$			
24. $y = \dfrac{3x^2 - 5}{7}$			

In Exercises 25–38, find the derivative of the algebraic function.

25. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$
26. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$
27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
28. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
29. $f(x) = \dfrac{2x + 5}{\sqrt{x}}$
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
31. $h(s) = (s^3 - 2)^2$
32. $h(x) = (x^2 - 1)^2$
33. $f(x) = \dfrac{2 - \frac{1}{x}}{x - 3}$
34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$
36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$
37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$, c is a constant
38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$, c is a constant

In Exercises 39–54, find the derivative of the trigonometric function.

39. $f(t) = t^2 \sin t$
40. $f(\theta) = (\theta + 1) \cos \theta$
41. $f(t) = \dfrac{\cos t}{t}$
42. $f(x) = \dfrac{\sin x}{x}$
43. $f(x) = -x + \tan x$
44. $y = x + \cot x$
45. $g(t) = \sqrt[4]{t} + 8 \sec t$
46. $h(s) = \dfrac{1}{s} - 10 \csc s$
47. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
48. $y = \dfrac{\sec x}{x}$
49. $y = -\csc x - \sin x$
50. $y = x \sin x + \cos x$
51. $f(x) = x^2 \tan x$
52. $f(x) = \sin x \cos x$
53. $y = 2x \sin x + x^2 \cos x$
54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

In Exercises 55–58, use a computer algebra system to differentiate the function.

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$
57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$
58. $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

In Exercises 59–62, evaluate the derivative of the function at the indicated point. Use a graphing utility to verify your result.

Function	*Point*
59. $y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62. $f(x) = \sin x(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
63. $f(x) = (x^3 - 3x + 1)(x + 2)$	$(1, -3)$
64. $f(x) = (x - 1)(x^2 - 2)$	$(0, 2)$
65. $f(x) = \dfrac{x}{x - 1}$	$(2, 2)$
66. $f(x) = \dfrac{(x - 1)}{(x + 1)}$	$\left(2, \dfrac{1}{3}\right)$
67. $f(x) = \tan x$	$\left(\dfrac{\pi}{4}, 1\right)$
68. $f(x) = \sec x$	$\left(\dfrac{\pi}{3}, 2\right)$

In Exercises 69 and 70, determine the point(s) at which the graph of the function has a horizontal tangent.

69. $f(x) = \dfrac{x^2}{x - 1}$

70. $f(x) = \dfrac{x^2}{x^2 + 1}$

In Exercises 71 and 72, verify that $f'(x) = g'(x)$, and explain the relationship between f and g.

71. $f(x) = \dfrac{3x}{x + 2}, \; g(x) = \dfrac{5x + 4}{x + 2}$

72. $f(x) = \dfrac{\sin x - 3x}{x}, \; g(x) = \dfrac{\sin x + 2x}{x}$

In Exercises 73 and 74, find the derivative of the function f for $n = 1, 2, 3,$ and 4. Use the result to write a general rule for $f'(x)$ in terms of n.

73. $f(x) = x^n \sin x$

74. $f(x) = \dfrac{\cos x}{x^n}$

75. ***Area*** The length of a rectangle is given by $2t + 1$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

76. ***Volume*** The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

77. ***Inventory Replenishment*** The ordering and transportation cost C for the components used in manufacturing a certain product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

78. ***Boyle's Law*** This law states that if the temperature of a gas remains constant, its pressure is inversely proportional to its volume. Use the derivative to show that the rate of change of the pressure is inversely proportional to the square of the volume.

79. ***Population Growth*** A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

80. ***Rate of Change*** Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

81. Prove the following differentiation rules.

(a) $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$

82. ***Modeling Data*** The table shows the number of motor homes n (in thousands) in the United States and the retail value v (in millions of dollars) of these motor homes for the years 1992 through 1997. The year is represented by t, with $t = 2$ corresponding to 1992. *(Source: Recreation Vehicle Industry Association)*

Year	1992	1993	1994	1995	1996	1997
n	226.3	243.8	306.7	281.0	274.6	239.3
v	\$6963	\$7544	\$9897	\$9768	\$9788	\$9139

(a) Use a graphing utility to find quadratic models for the number of motor homes $n(t)$ and the total retail value $v(t)$ of the motor homes.

(b) Find $A = v(t)/n(t)$. What does this function represent?

(c) Find $A'(t)$. Interpret the derivative in the context of these data.

In Exercises 83–88, find the second derivative of the function.

83. $f(x) = 4x^{3/2}$

84. $f(x) = x + 32x^{-2}$

85. $f(x) = \dfrac{x}{x - 1}$

86. $f(x) = \dfrac{x^2 + 2x - 1}{x}$

87. $f(x) = 3 \sin x$

88. $f(x) = \sec x$

In Exercises 89–92, find the higher-order derivative.

Given	*Find*
89. $f'(x) = x^2$	$f''(x)$
90. $f''(x) = 2 - \dfrac{2}{x}$	$f'''(x)$
91. $f'''(x) = 2\sqrt{x}$	$f^{(4)}(x)$
92. $f^{(4)}(x) = 2x + 1$	$f^{(6)}(x)$

Getting at the Concept

93. Sketch the graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$.

94. Sketch the graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x.

In Exercises 95–98, find $f'(2)$ given the following.

$g(2) = 3$ **and** $g'(2) = -2$

$h(2) = -1$ **and** $h'(2) = 4$

95. $f(x) = 2g(x) + h(x)$

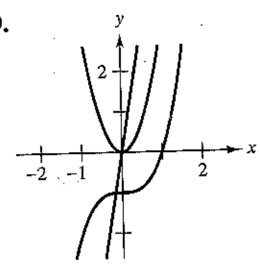
96. $f(x) = 4 - h(x)$

97. $f(x) = \dfrac{g(x)}{h(x)}$

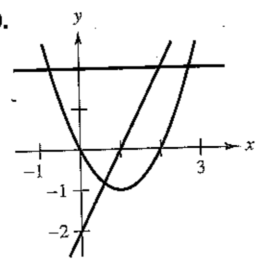
98. $f(x) = g(x)h(x)$

In Exercises 99 and 100, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

99.



100.



101. *Acceleration* The velocity of an object in meters per second is

$v(t) = 36 - t^2, \quad 0 \le t \le 6.$

Find the velocity and acceleration of the object when $t = 3$. What can be said about the speed of the object when the velocity and acceleration have opposite signs?

102. *Stopping Distance* A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is

$s(t) = -8.25t^2 + 66t$

where s is measured in feet and t is measured in seconds. Use this function to complete the table, and find the average velocity during each time interval.

t	0	1	2	3	4
$s(t)$					
$v(t)$					
$a(t)$					

103. *Acceleration* An automobile's velocity starting from rest is

$$v(t) = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at each of the following times.

(a) 5 seconds (b) 10 seconds (c) 20 seconds

104. *Finding a Pattern* Develop a general rule for $f^{(n)}(x)$ if

(a) $f(x) = x^n$ and (b) $f(x) = \dfrac{1}{x}$.

105. *Finding a Pattern* Consider the function $f(x) = g(x)h(x)$.

(a) Use the product rule to generate rules for finding $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

(b) Use the results in part (a) to write a general rule for $f^{(n)}(x)$.

106. *Finding a Pattern* Develop a general rule for $[xf(x)]^{(n)}$ where f is a differentiable function of x.

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$P_1(x) = f'(a)(x - a) + f(a)$ **and**

$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$

In Exercises 107 and 108, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

107. $f(x) = \cos x$, $a = \dfrac{\pi}{3}$

108. $f(x) = \sin x$, $a = \dfrac{\pi}{2}$

True or False? **In Exercises 109–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

109. If $y = f(x)g(x)$, then $dy/dx = f'(x)g'(x)$.

110. If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $d^5y/dx^5 = 0$.

111. If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

112. If $f(x)$ is an nth-degree polynomial, then $f^{(n+1)}(x) = 0$.

113. The second derivative represents the rate of change of the first derivative.

114. If the velocity of an object is constant, then its acceleration is zero.

115. Find the derivative of $f(x) = x|x|$. Does $f''(0)$ exist?

116. *Think About It* Let f and g be functions whose first and second derivatives exist on an interval I. Which of the following formulas is (are) true?

(a) $fg'' - f''g = (fg' - f'g)'$

(b) $fg'' + f''g = (fg)''$

We conclude this section with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule.

Summary of Differentiation Rules

General Differentiation Rules Let f, g, and u be differentiable functions of x.

Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$

Sum or Difference Rule: $\frac{d}{dx}[f \pm g] = f' \pm g'$

Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$

Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

Derivatives of Algebraic Functions

Constant Rule: $\frac{d}{dx}[c] = 0$

(Simple) Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Chain Rule

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\,u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}u'$

STUDY TIP As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

EXERCISES FOR SECTION 2.4

In Exercises 1–6, complete the table using Example 2 as a model.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$		
2. $y = \frac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^2 - 1}$		
4. $y = 3\tan(\pi x^2)$		
5. $y = \csc^3 x$		
6. $y = \cos\frac{3x}{2}$		

In Exercises 7–34, find the derivative of the function.

7. $y = (2x - 7)^3$
8. $y = (2x^3 + 1)^2$
9. $g(x) = 3(4 - 9x)^4$
10. $y = 3(4 - x^2)^5$
11. $f(x) = (9 - x^2)^{2/3}$
12. $f(t) = (9t + 2)^{2/3}$
13. $f(t) = \sqrt{1 - t}$
14. $g(x) = \sqrt{5 - 3x}$
15. $y = \sqrt[3]{9x^2 + 4}$
16. $g(x) = \sqrt{x^2 - 2x + 1}$
17. $y = 2\sqrt[4]{4 - x^2}$
18. $f(x) = -3\sqrt[4]{2 - 9x}$
19. $y = \frac{1}{x - 2}$
20. $s(t) = \frac{1}{t^2 + 3t - 1}$
21. $f(t) = \left(\frac{1}{t - 3}\right)^2$
22. $y = -\frac{5}{(t + 3)^3}$
23. $y = \frac{1}{\sqrt{x + 2}}$
24. $g(t) = \sqrt{\frac{1}{t^2 - 2}}$
25. $f(x) = x^2(x - 2)^4$
26. $f(x) = x(3x - 9)^3$
27. $y = x\sqrt{1 - x^2}$
28. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$
29. $y = \frac{x}{\sqrt{x^2 + 1}}$
30. $y = \frac{x}{\sqrt{x^4 + 4}}$
31. $g(x) = \left(\frac{x + 5}{x^2 + 2}\right)^2$
32. $h(t) = \left(\frac{t^2}{t^3 + 2}\right)^2$
33. $f(v) = \left(\frac{1 - 2v}{1 + v}\right)^3$
34. $g(x) = \left(\frac{3x^2 - 2}{2x + 3}\right)^3$

In Exercises 35–44, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

35. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

36. $y = \sqrt{\dfrac{2x}{x + 1}}$

37. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

38. $f(x) = \sqrt{x}(2 - x)^2$

39. $y = \sqrt{\dfrac{x + 1}{x}}$

40. $y = (t^2 - 9)\sqrt{t + 2}$

41. $s(t) = \dfrac{-2(2 - t)\sqrt{1 + t}}{3}$

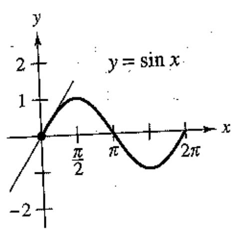
42. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

43. $y = \dfrac{\cos \pi x + 1}{x}$

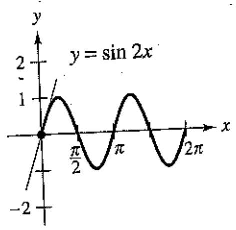
44. $y = x^2 \tan \dfrac{1}{x}$

In Exercises 45 and 46, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

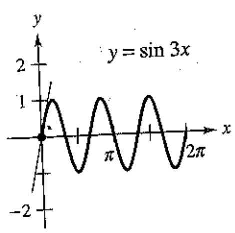
45. (a)



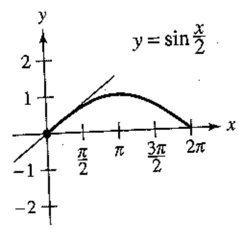
(b)



46. (a)



(b)



In Exercises 47–66, find the derivative of the function.

47. $y = \cos 3x$

48. $y = \sin \pi x$

49. $g(x) = 3 \tan 4x$

50. $h(x) = \sec x^2$

51. $y = \sin(\pi x)^2$

52. $y = \cos(1 - 2x)^2$

53. $h(x) = \sin 2x \cos 2x$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)$

55. $f(x) = \dfrac{\cot x}{\sin x}$

56. $g(v) = \dfrac{\cos v}{\csc v}$

57. $y = 4 \sec^2 x$

58. $y = 2 \tan^3 x$

59. $f(\theta) = \frac{1}{4} \sin^2 2\theta$

60. $g(t) = 5 \cos^2 \pi t$

61. $f(t) = 3 \sec^2(\pi t - 1)$

62. $h(t) = 2 \cot^2(\pi t + 2)$

63. $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

64. $y = 3x - 5 \cos(\pi x)^2$

65. $y = \sin(\cos x)$

66. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

In Exercises 67–74, evaluate the derivative of the function at the indicated point. Use a graphing utility to verify your result.

	Function	*Point*
67.	$s(t) = \sqrt{t^2 + 2t + 8}$	$(2, 4)$
68.	$y = \sqrt[5]{3x^3 + 4x}$	$(2, 2)$
69.	$f(x) = \dfrac{3}{x^3 - 4}$	$\left(-1, -\frac{3}{5}\right)$
70.	$f(x) = \dfrac{1}{(x^2 - 3x)^2}$	$\left(4, \frac{1}{16}\right)$
71.	$f(t) = \dfrac{3t + 2}{t - 1}$	$(0, -2)$
72.	$f(x) = \dfrac{x + 1}{2x - 3}$	$(2, 3)$
73.	$y = 37 - \sec^3(2x)$	$(0, 36)$
74.	$y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

In Exercises 75–78, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

	Function	*Point*
75.	$f(x) = \sqrt{3x^2 - 2}$	$(3, 5)$
76.	$f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$	$(2, 2)$
77.	$f(x) = \sin 2x$	$(\pi, 0)$
78.	$f(x) = \tan^2 x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 79–82, find the second derivative of the function.

79. $f(x) = 2(x^2 - 1)^3$

80. $f(x) = \dfrac{1}{x - 2}$

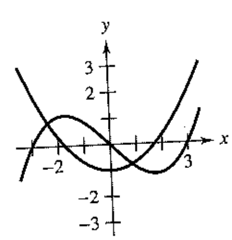
81. $f(x) = \sin x^2$

82. $f(x) = \sec^2 \pi x$

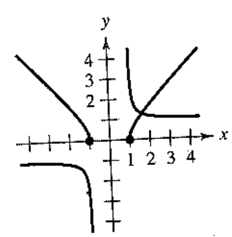
Getting at the Concept

In Exercises 83–86, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

83.

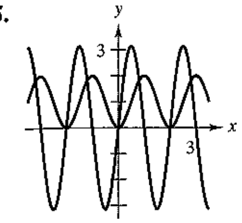


84.

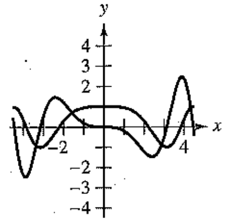


Getting at the Concept *(continued)*

85.



86.



In Exercises 87 and 88, the relationship between f and g is given. State the relationship between f' and g'.

87. $g(x) = f(3x)$

88. $g(x) = f(x^2)$

89. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

90. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that $f'(x) = g'(x)$.

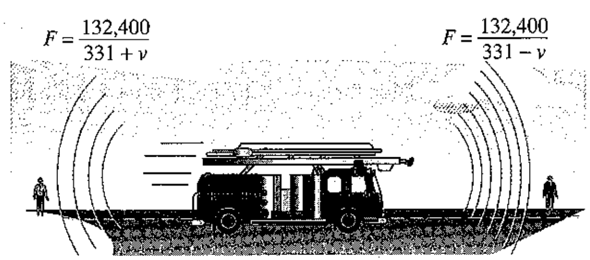
91. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



92. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

93. ***Pendulum*** A 15-centimeter pendulum moves according to the equation

$$\theta = 0.2 \cos 8t$$

where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

94. ***Wave Motion*** A buoy oscillates in simple harmonic motion

$$y = A \cos \omega t$$

as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

95. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

96. ***Modeling Data*** The normal daily maximum temperature T (in degrees Fahrenheit) for Denver, Colorado, is shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	43.2	46.6	52.2	61.8	70.8	81.4

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	88.2	85.8	76.9	66.3	52.5	44.5

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(\pi t/6 - c)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

97. ***Modeling Data*** The cost of producing x units of a product is $C = 60x + 1350$. For one week management determined the number of units produced at the end of t hours during an 8-hour shift. The average values of x for the week are shown in the table.

t	0	1	2	3	4	5	6	7	8
x	0	16	60	130	205	271	336	384	392

(a) Use a graphing utility to fit a cubic model to the data.

(b) Use the Chain Rule to find dC/dt.

(c) Explain why the cost function is not increasing at a constant rate during the 8-hour shift.

98. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

99. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results in part (a) to write general rules for the even- and odd-order derivatives

$$f^{(2k)}(x) \text{ and } f^{(2k-1)}(x).$$

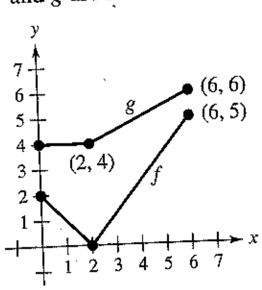
[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

100. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

101. ***Think About It*** Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$ where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



102. Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

103. The geometric mean of x and $x + n$ is $g = \sqrt{x(x + n)}$, and the arithmetic mean is $a = [x + (x + n)]/2$. Show that

$$\frac{dg}{dx} = \frac{a}{g}.$$

104. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 105–108, use the result of Exercise 104 to find the derivative of the function.

105. $g(x) = |2x - 3|$

106. $f(x) = |x^2 - 4|$

107. $h(x) = |x| \cos x$

108. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$
$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 109 and 110, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

109. $f(x) = \tan \dfrac{\pi x}{4}$, $a = 1$

110. $f(x) = \sec 2x$, $a = \dfrac{\pi}{6}$

True or False? **In Exercises 111–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

111. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

112. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

113. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

114. You would first apply the General Power Rule to find the derivative of $y = x \sin^3 x$.

EXERCISES FOR SECTION 2.5

In Exercises 1–16, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 36$
2. $x^2 - y^2 = 16$
3. $x^{1/2} + y^{1/2} = 9$
4. $x^3 + y^3 = 8$
5. $x^3 - xy + y^2 = 4$
6. $x^2y + y^2x = -2$
7. $x^3y^3 - y = x$
8. $\sqrt{xy} = x - 2y$
9. $x^3 - 3x^2y + 2xy^2 = 12$
10. $2 \sin x \cos y = 1$
11. $\sin x + 2 \cos 2y = 1$
12. $(\sin \pi x + \cos \pi y)^2 = 2$
13. $\sin x = x(1 + \tan y)$
14. $\cot y = x - y$
15. $y = \sin(xy)$
16. $x = \sec \dfrac{1}{y}$

In Exercises 17–20, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).

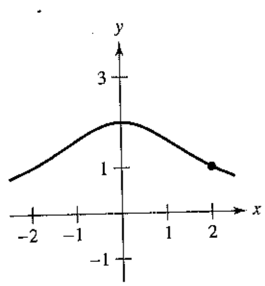
17. $x^2 + y^2 = 16$
18. $x^2 + y^2 - 4x + 6y + 9 = 0$
19. $9x^2 + 16y^2 = 144$
20. $9y^2 - x^2 = 9$

In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the indicated point.

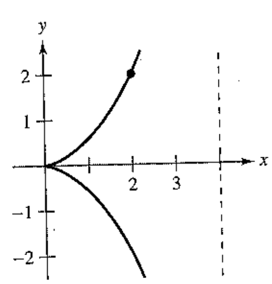
	Equation	*Point*
21.	$xy = 4$	$(-4, -1)$
22.	$x^2 - y^3 = 0$	$(1, 1)$
23.	$y^2 = \dfrac{x^2 - 4}{x^2 + 4}$	$(2, 0)$
24.	$(x + y)^3 = x^3 + y^3$	$(-1, 1)$
25.	$x^{2/3} + y^{2/3} = 5$	$(8, 1)$
26.	$x^3 + y^3 = 4xy + 1$	$(2, 1)$
27.	$\tan(x + y) = x$	$(0, 0)$
28.	$x \cos y = 1$	$\left(2, \dfrac{\pi}{3}\right)$

In Exercises 29–32, find the slope of the tangent line to the graph at the indicated point.

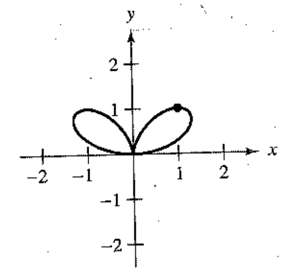
29. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



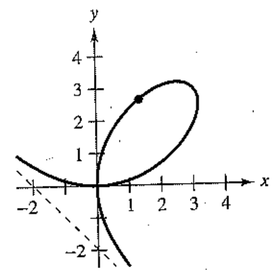
30. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



31. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$



32. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$



In Exercises 33 and 34, find dy/dx implicitly and find the largest interval of the form $-a < y < a$ such that y is a differentiable function of x. Then express dy/dx as a function of x.

33. $\tan y = x$
34. $\cos y = x$

In Exercises 35–40, find d^2y/dx^2 in terms of x and y.

35. $x^2 + y^2 = 36$
36. $x^2y^2 - 2x = 3$
37. $x^2 - y^2 = 16$
38. $1 - xy = x - y$
39. $y^2 = x^3$
40. $y^2 = 4x$

In Exercises 41 and 42, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the indicated point and sketch its graph.

41. $\sqrt{x} + \sqrt{y} = 4, \quad (9, 1)$
42. $y^2 = \dfrac{x - 1}{x^2 + 1}, \quad \left(2, \dfrac{\sqrt{5}}{5}\right)$

In Exercises 43 and 44, find equations for the tangent line and normal line to the circle at the indicated points. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.

43. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$
44. $x^2 + y^2 = 9$
$(0, 3), (2, \sqrt{5})$

45. Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.
46. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

In Exercises 47 and 48, find the points at which the graph of the equation has a vertical or horizontal tangent line.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$
48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

Orthogonal Trajectories **In Exercises 49–52, use a graphing utility to sketch the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection their tangent lines are perpendicular to each other.]**

49. $2x^2 + y^2 = 6$
$y^2 = 4x$

50. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

51. $x + y = 0$
$x = \sin y$

52. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 53 and 54, verify that the two families of curves are orthogonal where C and K are real numbers. Use a graphing utility to graph the two families for two values of C and two values of K.**

53. $xy = C$
$x^2 - y^2 = K$

54. $x^2 + y^2 = C^2$
$y = Kx$

In Exercises 55–58, differentiate (a) with respect to x (y is a function of x) and (b) with respect to t (x and y are functions of t).

55. $2y^2 - 3x^4 = 0$

56. $x^2 - 3xy^2 + y^3 = 10$

57. $\cos \pi y - 3 \sin \pi x = 1$

58. $4 \sin x \cos y = 1$

Getting at the Concept

59. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

60. In your own words, state the guidelines for implicit differentiation.

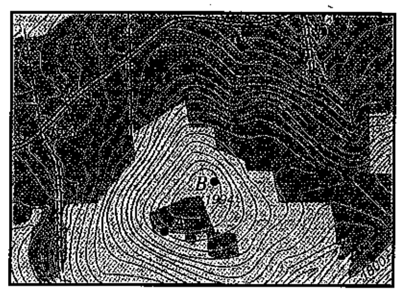
61. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

62. ***Orthogonal Trajectories*** The figure below gives the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



63. Prove (Theorem 2.3) that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

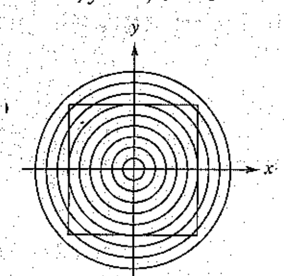
for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

64. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

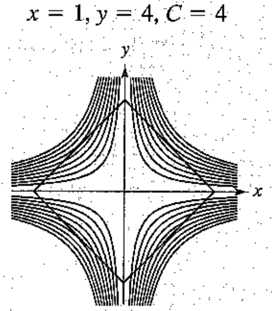
SECTION PROJECT OPTICAL ILLUSIONS

In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of dy/dx for the indicated values of x and y.

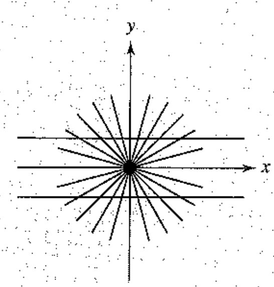
(a) Circles: $x^2 + y^2 = C^2$
$x = 3, y = 4, C = 5$



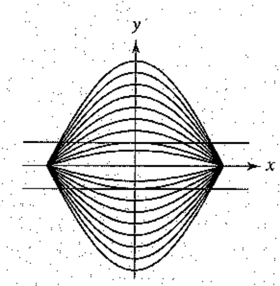
(b) Hyperbolas: $xy = C$
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$
$x = \sqrt{3}, y = 3,$
$a = \sqrt{3}, b = 1$



(d) Cosine curves: $y = C \cos x$
$x = \frac{\pi}{3}, y = \frac{1}{3}, C = \frac{2}{3}$



FOR FURTHER INFORMATION For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*. To view this article, go to the website *www.matharticles.com*.

EXERCISES FOR SECTION 2.6

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

Equation	*Find*	*Given*
1. $y = \sqrt{x}$	(a) $\dfrac{dy}{dt}$ when $x = 4$	$\dfrac{dx}{dt} = 3$
	(b) $\dfrac{dx}{dt}$ when $x = 25$	$\dfrac{dy}{dt} = 2$
2. $y = 2(x^2 - 3x)$	(a) $\dfrac{dy}{dt}$ when $x = 3$	$\dfrac{dx}{dt} = 2$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = 5$
3. $xy = 4$	(a) $\dfrac{dy}{dt}$ when $x = 8$	$\dfrac{dx}{dt} = 10$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = -6$
4. $x^2 + y^2 = 25$	(a) $\dfrac{dy}{dt}$ when $x = 3, y = 4$	$\dfrac{dx}{dt} = 8$
	(b) $\dfrac{dx}{dt}$ when $x = 4, y = 3$	$\dfrac{dy}{dt} = -2$

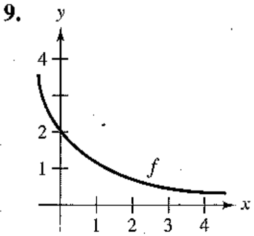
In Exercises 5–8, a point is moving along the graph of the function such that dx/dt is 2 centimeters per second. Find dy/dt for the specified values of x.

Function	*Values of x*		
5. $y = x^2 + 1$	(a) $x = -1$	(b) $x = 0$	(c) $x = 1$
6. $y = \dfrac{1}{1 + x^2}$	(a) $x = -2$	(b) $x = 0$	(c) $x = 2$
7. $y = \tan x$	(a) $x = -\dfrac{\pi}{3}$	(b) $x = -\dfrac{\pi}{4}$	(c) $x = 0$
8. $y = \sin x$	(a) $x = \dfrac{\pi}{6}$	(b) $x = \dfrac{\pi}{4}$	(c) $x = \dfrac{\pi}{3}$

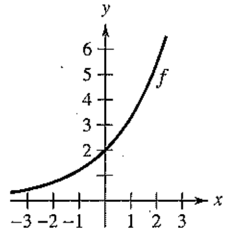
Getting at the Concept

In Exercises 9 and 10, using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive.

9.



10.



11. Consider the linear function $y = ax + b$. If x changes at a constant rate, does y change at a constant rate? If so, does it change at the same rate as x? Explain.

12. In your own words, state the guidelines for solving related rate problems.

13. Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $dx/dt = 2$ centimeters per second.

14. Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $dx/dt = 2$ centimeters per second.

15. ***Area*** The radius r of a circle is increasing at a rate of 3 centimeters per minute. Find the rate of change of the area when (a) $r = 6$ centimeters and (b) $r = 24$ centimeters.

16. ***Area*** Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

17. ***Area*** The included angle of the two sides of constant equal length s of an isosceles triangle is θ.
    (a) Show that the area of the triangle is given by $A = \frac{1}{2}s^2 \sin \theta$.
    (b) If θ is increasing at the rate of $\frac{1}{2}$ radian per minute, find the rate of change of the area when $\theta = \pi/6$ and $\theta = \pi/3$.
    (c) Explain why the rate of change of the area of the triangle is not constant even though $d\theta/dt$ is constant.

18. ***Volume*** The radius r of a sphere is increasing at a rate of 2 inches per minute.
    (a) Find the rate of change of the volume when $r = 6$ inches and $r = 24$ inches.
    (b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

19. ***Volume*** A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 30 centimeters and (b) 60 centimeters?

20. ***Volume*** All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

21. ***Surface Area*** The conditions are the same as in Exercise 20. Determine how fast the *surface area* is changing when each edge is (a) 1 centimeter and (b) 10 centimeters.

22. ***Volume*** The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

23. ***Volume*** At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high?

24. ***Depth*** A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

25. ***Depth*** A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end (see figure). Water is being pumped into the pool at $\frac{1}{4}$ cubic meter per minute, and there is 1 meter of water at the deep end.

(a) What percent of the pool is filled?

(b) At what rate is the water level rising?

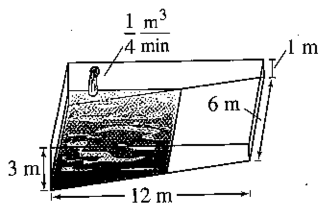


Figure for 25

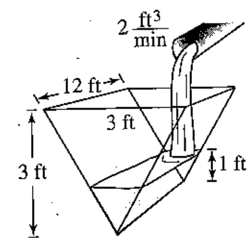


Figure for 26

26. ***Depth*** A trough is 12 feet long and 3 feet across the top (see figure). It's ends are isosceles triangles with altitudes of 3 feet.

(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when it is 1 foot deep?

(b) If the water is rising at a rate of $\frac{3}{8}$ inch per minute when $h = 2$, determine the rate at which water is being pumped into the trough.

27. ***Moving Ladder*** A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top moving down the wall when the base of the ladder is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

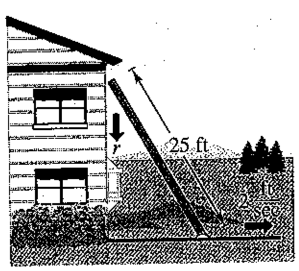


Figure for 27

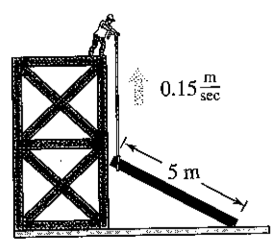


Figure for 28

FOR FURTHER INFORMATION For more information on the mathematics of moving ladders, see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com.*

28. ***Construction*** A construction worker pulls a 5-meter plank up the side of a building under construction by means of a rope tied to one end of the plank (see figure). Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meter per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

29. ***Construction*** A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position, as shown in the figure. The winch pulls in rope at a rate of -0.2 meter per second. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$.

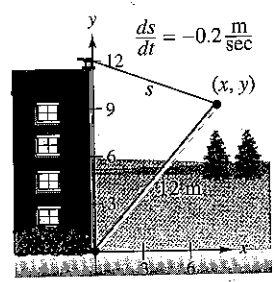


Figure for 29

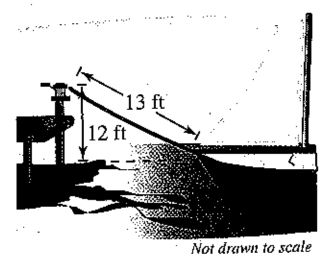


Figure for 30

30. ***Boating*** A boat is pulled into a dock by means of a winch 12 feet above the deck of the boat (see figure).

(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?

(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls in rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

31. ***Air Traffic Control*** An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other (see figure). One plane is 150 miles from the point moving at 450 miles per hour. The other plane is 200 miles from the point moving at 600 miles per hour.

(a) At what rate is the distance between the planes decreasing?

(b) How much time does the air traffic controller have to get one of the planes on a different flight path?

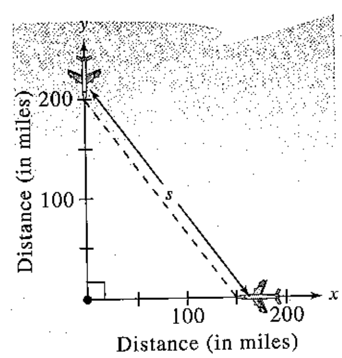


Figure for 31

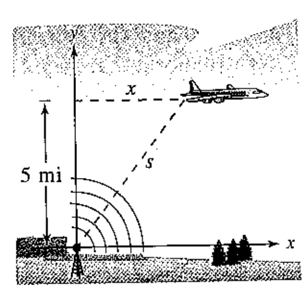


Figure for 32

32. ***Air Traffic Control*** An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

33. ***Baseball*** A baseball diamond has the shape of a square with sides 90 feet long (see figure). A player running from second base to third base at a speed of 28 feet per second is 30 feet from third base. At what rate is the player's distance s from home plate changing?

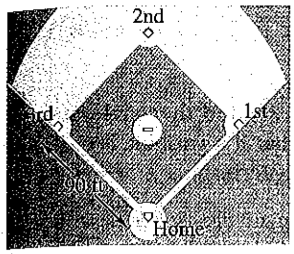


Figure for 33 and 34

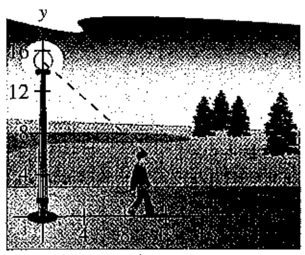


Figure for 35

34. ***Baseball*** For the baseball diamond in Exercise 33, suppose the player is running from first to second at a speed of 28 feet per second. Find the rate at which the distance from home plate is changing when the player is 30 feet from second base.

35. ***Shadow Length*** A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

36. ***Shadow Length*** Repeat Exercise 35 for a man 6 feet tall walking at a rate of 5 feet per second *toward* a light that is 20 feet above the ground (see figure).

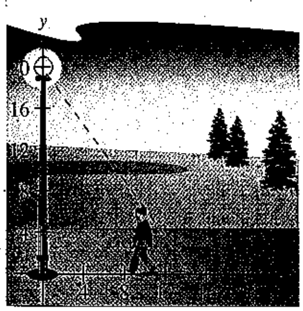


Figure for 36

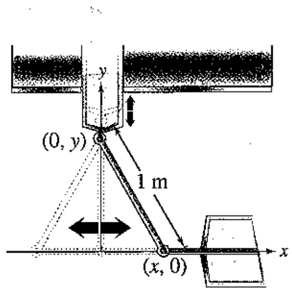


Figure for 37

37. ***Machine Design*** The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$ (see figure). The position of the end on the x-axis is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$$

where t is the time in seconds.

(a) Find the time of one complete cycle of the rod.

(b) What is the lowest point reached by the end of the rod on the y-axis?

(c) Find the speed of the y-axis endpoint when the x-axis endpoint is $\left(\frac{1}{4}, 0\right)$.

38. ***Machine Design*** Repeat Exercise 37 for a position function of $x(t) = \frac{3}{5}\sin \pi t$. Use the point $\left(\frac{3}{10}, 0\right)$ for part (c).

39. ***Evaporation*** As a spherical raindrop falls, it reaches a layer of dry air and begins to evaporate at a rate that is proportional to its surface area $(S = 4\pi r^2)$. Show that the radius of the raindrop decreases at a constant rate.

40. ***Electricity*** The combined electrical resistance R of R_1 and R_2, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R, R_1, and R_2 are measured in ohms. R_1 and R_2 are increasing at rates of 1 and 1.5 ohms per second, respectively. At what rate is R changing when $R_1 = 50$ ohms and $R_2 = 75$ ohms?

41. ***Adiabatic Expansion*** When a certain polyatomic gas undergoes adiabatic expansion, its pressure p and volume V satisfy the equation

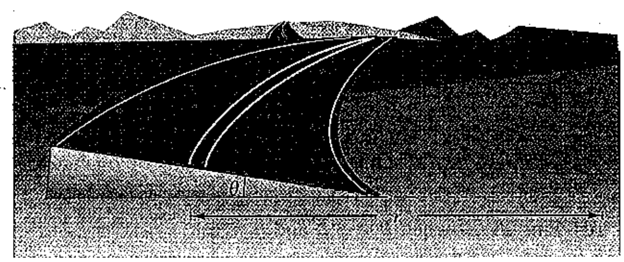
$$pV^{1.3} = k$$

where k is a constant. Find the relationship between the related rates dp/dt and dV/dt.

42. ***Roadway Design*** Cars on a certain roadway travel on a circular arc of radius r. In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal (see figure). The banking angle must satisfy the equation

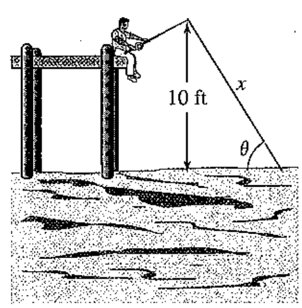
$$rg \tan \theta = v^2$$

where v is the velocity of the cars and $g = 32$ feet per second per second is the acceleration due to gravity. Find the relationship between the related rates dv/dt and $d\theta/dt$.

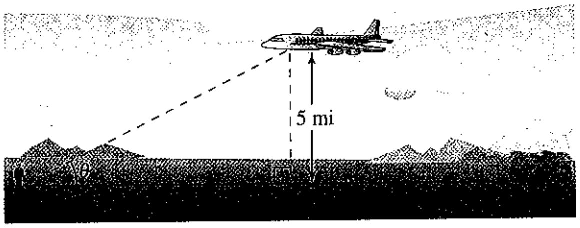


43. ***Angle of Elevation*** A balloon rises at a rate of 3 meters per second from a point on the ground 30 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground.

44. ***Angle of Elevation*** A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water (see figure). At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?



45. ***Angle of Elevation*** An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure). The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation θ is changing when the angle is (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 75°$.



46. ***Linear vs. Angular Speed*** A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of (a) $\theta = 30°$, (b) $\theta = 60°$, and (c) $\theta = 70°$ with the line perpendicular from the light to the wall?

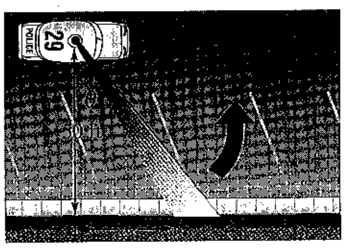


Figure for 46

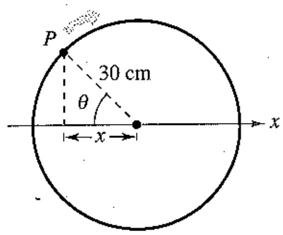


Figure for 47

47. ***Linear vs. Angular Speed*** A wheel of radius 30 centimeters revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel (see figure).

(a) Find dx/dt as a function of θ.

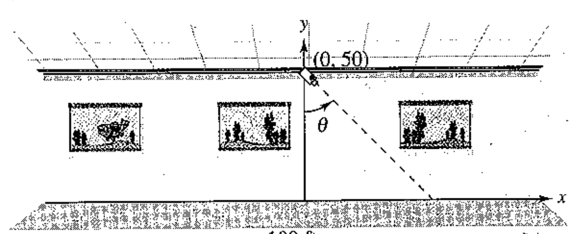
(b) Use a graphing utility to graph the function in part (a).

(c) When is the absolute value of the rate of change of x greatest? When is it least?

(d) Find dx/dt when $\theta = 30°$ and $\theta = 60°$.

48. ***Flight Control*** An airplane is flying in still air with an airspeed of 240 miles per hour. If it is climbing at an angle of 22°, find the rate at which it is gaining altitude.

49. ***Security Camera*** A security camera is centered 50 feet above a 100-foot hallway (see figure). It is easiest to design the camera with a constant angular rate of rotation, but this results in a variable rate at which the images of the surveillance area are recorded. Therefore, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation if $|dx/dt| = 2$ feet per second.



50. ***Think About It*** Describe the relationship between the rate of change of y and the rate of change of x in each of the following. Assume all variables and derivatives are positive.

(a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$

(b) $\dfrac{dy}{dt} = x(L - x)\dfrac{dx}{dt}, \quad 0 \le x \le L$

Acceleration **In Exercises 51 and 52, find the acceleration of the specified object. (*Hint:* Recall that if a variable is changing at a constant rate, its acceleration is zero.)**

51. Find the acceleration of the top of the ladder described in Exercise 27 when the base of the ladder is 7 feet from the wall.

52. Find the acceleration of the boat in Exercise 30(a) when there is a total of 13 feet of rope out.

53. ***Modeling Data*** The table shows the numbers (in millions) of single women s and married women m in the civilian work force in the United States for the years 1990 through 1998. *(Source: U.S. Bureau of Labor Statistics)*

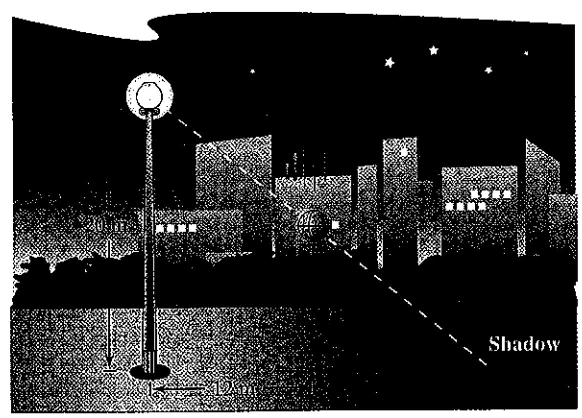
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998
s	14.6	14.7	14.9	15.0	15.3	15.5	15.8	16.5	17.1
m	30.9	31.1	31.7	32.0	32.9	33.4	33.6	33.8	33.9

(a) Use the regression capabilities of a graphing utility to find a model of the form $m(s) = as^2 + bs + c$ for the data, where t is the time in years, with $t = 0$ corresponding to 1990.

(b) Find $\dfrac{dm}{dt}$.

(c) Use the model to estimate dm/dt for $t = 5$ if it is predicted that the number of single women in the work force will increase at the rate of 1.2 million per year.

54. A ball is dropped from a height of 20 meters, 12 meters away from the top of a 20-meter lamppost (see figure). The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released? *(Submitted by Dennis Gittinger, St. Philips College, San Antonio, TX)*



Lab Series LAB 4

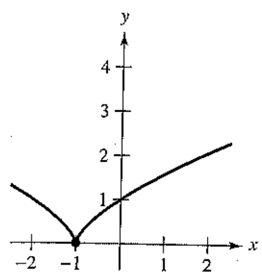
REVIEW EXERCISES FOR CHAPTER 2

2.1 **In Exercises 1–4, find the derivative of the function by using the definition of the derivative.**

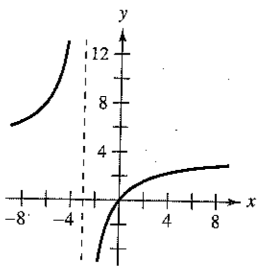
1. $f(x) = x^2 - 2x + 3$
2. $f(x) = \dfrac{x+1}{x-1}$
3. $f(x) = \sqrt{x} + 1$
4. $f(x) = \dfrac{2}{x}$

In Exercises 5 and 6, describe the x-values at which f is differentiable.

5. $f(x) = (x+1)^{2/3}$



6. $f(x) = \dfrac{4x}{x+3}$



7. Sketch the graph of $f(x) = 4 - |x - 2|$.
   (a) Is f continuous at $x = 2$?
   (b) Is f differentiable at $x = 2$? Explain.
8. Sketch the graph of $f(x) = \begin{cases} x^2 + 4x + 2, & x < -2 \\ 1 - 4x - x^2, & x \geq -2. \end{cases}$
   (a) Is f continuous at $x = -2$?
   (b) Is f differentiable at $x = -2$? Explain.

In Exercises 9 and 10, find the slope of the tangent line to the graph of the function at the specified point.

9. $g(x) = \dfrac{2}{3}x^2 - \dfrac{x}{6}, \quad \left(-1, \dfrac{5}{6}\right)$
10. $h(x) = \dfrac{3x}{8} - 2x^2, \quad \left(-2, -\dfrac{35}{4}\right)$

In Exercises 11 and 12, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

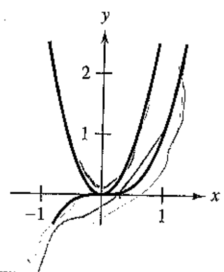
11. $f(x) = x^3 - 1, \quad (-1, -2)$
12. $f(x) = \dfrac{2}{x+1}, \quad (0, 2)$

In Exercises 13 and 14, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

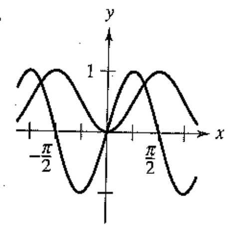
13. $g(x) = x^2(x-1), \quad c = 2$
14. $f(x) = \dfrac{1}{x+1}, \quad c = 2$

Writing **In Exercises 15 and 16, the figure shows the graphs of a function and its derivative. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15.



16.



2.2 **In Exercises 17–32, find the derivative of the function.**

17. $y = 25$
18. $y = -12$
19. $f(x) = x^8$
20. $g(x) = x^{12}$
21. $h(t) = 3t^4$
22. $f(t) = -8t^5$
23. $f(x) = x^3 - 3x^2$
24. $g(s) = 4s^4 - 5s^2$
25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x}$
26. $f(x) = x^{1/2} - x^{-1/2}$
27. $g(t) = \dfrac{2}{3t^2}$
28. $h(x) = \dfrac{2}{(3x)^2}$
29. $f(\theta) = 2\theta - 3\sin\theta$
30. $g(\alpha) = 4\cos\alpha + 6$
31. $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$
32. $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

33. ***Vibrating String*** When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds. Find the rate of change of F when (a) $T = 4$ and (b) $T = 9$.
34. ***Vertical Motion*** A ball is dropped from a height of 100 feet. One second later, another ball is dropped from a height of 75 feet. Which ball hits the ground first?
35. ***Vertical Motion*** To estimate the height of a building, a weight is dropped from the top of the building into a pool at ground level. How high is the building if the splash is seen 9.2 seconds after the weight is dropped?
36. ***Vertical Motion*** A bomb is dropped from an airplane at an altitude of 14,400 feet. How long will it take for the bomb to reach the ground? (Because of the motion of the plane, the fall will not be vertical, but the time will be the same as that for a vertical fall.) The plane is moving at 600 miles per hour. How far will the bomb move horizontally after it is released from the plane?

37. ***Projectile Motion*** A ball thrown follows a path described by $y = x - 0.02x^2$.
   (a) Sketch a graph of the path.
   (b) Find the total horizontal distance the ball was thrown.
   (c) At what x-value does the ball reach its maximum height? (Use the symmetry of the path.)
   (d) Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate the equation at $x = 0, 10, 25, 30$, and 50.
   (e) What is the instantaneous rate of change of the height when the ball reaches its maximum height?

38. ***Projectile Motion*** The path of a projectile thrown at an angle of 45° with level ground is

$$y = x - \frac{32}{v_0^2}(x^2)$$

where the initial velocity is v_0 feet per second.
   (a) Find the x-coordinate of the point where the projectile strikes the ground. Use the symmetry of the path of the projectile to locate the x-coordinate of the point where the projectile reaches its maximum height.
   (b) What is the instantaneous rate of change of the height when the projectile is at its maximum height?
   (c) Show that doubling the initial velocity of the projectile multiplies both the maximum height and the range by a factor of 4.
   (d) Find the maximum height and range of a projectile thrown with an initial velocity of 70 feet per second. Use a graphing utility to sketch the path of the projectile.

39. ***Horizontal Motion*** The position function of a particle moving along the x-axis is

$$x(t) = t^2 - 3t + 2$$

for $-\infty < t < \infty$.
   (a) Find the velocity of the particle.
   (b) Find the open t-interval(s) in which the particle is moving to the left.
   (c) Find the position of the particle when the velocity is 0.
   (d) Find the speed of the particle when the position is 0.

40. ***Modeling Data*** The speed of a car in miles per hour and the stopping distance in feet are recorded in the table.

Speed (x)	20	30	40	50	60
Stopping Distance (y)	25	55	105	188	300

   (a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.
   (b) Use a graphing utility to plot the data and graph the model.
   (c) Use a graphing utility to graph dy/dx.
   (d) Use the model to approximate the stopping distance at a speed of 65 miles per hour.
   (e) Use the graphs in parts (b) and (c) to explain the change in stopping distance as the speed increases.

2.3 **In Exercises 41–57, find the derivative of the function.**

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$
42. $g(x) = (x^3 - 3x)(x + 2)$
43. $h(x) = \sqrt{x}\sin x$
44. $f(t) = t^3 \cos t$
45. $f(x) = \dfrac{2x^3 - 1}{x^2}$
46. $f(x) = \dfrac{x + 1}{x - 1}$
47. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$
48. $f(x) = \dfrac{6x - 5}{x^2 + 1}$
49. $f(x) = \dfrac{1}{4 - 3x^2}$
50. $f(x) = \dfrac{9}{3x^2 - 2x}$
51. $y = \dfrac{x^2}{\cos x}$
52. $y = \dfrac{\sin x}{x^2}$
53. $y = 3x^2 \sec x$
54. $y = 2x - x^2 \tan x$
55. $y = -x \tan x$
56. $y = \dfrac{1 + \sin x}{1 - \sin x}$
57. $y = x \cos x - \sin x$

58. ***Acceleration*** The velocity of an object in meters per second is $v(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when $t = 4$.

In Exercises 59–62, find the second derivative of the function.

59. $g(t) = t^3 - 3t + 2$
60. $f(x) = 12\sqrt[4]{x}$
61. $f(\theta) = 3 \tan \theta$
62. $h(t) = 4 \sin t - 5 \cos t$

In Exercises 63 and 64, show that the function satisfies the equation.

	Function	*Equation*
63.	$y = 2 \sin x + 3 \cos x$	$y'' + y = 0$
64.	$y = \dfrac{10 - \cos x}{x}$	$xy' + y = \sin x$

2.4 **In Exercises 65–80, find the derivative of the function.**

65. $f(x) = \sqrt{1 - x^3}$
66. $f(x) = \sqrt[3]{x^2 - 1}$
67. $h(x) = \left(\dfrac{x - 3}{x^2 + 1}\right)^2$
68. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$
69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$
70. $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$
71. $y = 3\cos(3x + 1)$
72. $y = 1 - \cos 2x + 2\cos^2 x$
73. $y = \frac{1}{2}\csc 2x$
74. $y = \csc 3x + \cot 3x$
75. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
76. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$
77. $y = \dfrac{2}{3}\sin^{3/2} x - \dfrac{2}{7}\sin^{7/2} x$
78. $f(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$
79. $y = \dfrac{\sin \pi x}{x + 2}$
80. $y = \dfrac{\cos(x - 1)}{x - 1}$

In Exercises 81–88, use a computer algebra system to find the derivative of the function. Use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

81. $f(t) = t^2(t-1)^5$

82. $f(x) = [(x-2)(x+4)]^2$

83. $g(x) = \dfrac{2x}{\sqrt{x+1}}$

84. $g(x) = x\sqrt{x^2+1}$

85. $f(t) = \sqrt{t+1}\sqrt[3]{t+1}$

86. $y = \sqrt{3x}(x+2)^3$

87. $y = \tan\sqrt{1-x}$

88. $y = 2\csc^3(\sqrt{x})$

In Exercises 89–92, find the second derivative of the function.

89. $y = 2x^2 + \sin 2x$

90. $y = \dfrac{1}{x} + \tan x$

91. $f(x) = \cot x$

92. $y = \sin^2 x$

In Exercises 93–96, use a computer algebra system to find the second derivative of the function.

93. $f(t) = \dfrac{t}{(1-t)^2}$

94. $g(x) = \dfrac{6x-5}{x^2+1}$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

96. $h(x) = x\sqrt{x^2-1}$

97. ***Refrigeration*** The temperature T of food put in a freezer is

$$T = \frac{700}{t^2 + 4t + 10}$$

where t is the time in hours. Find the rate of change of T with respect to t at each of the following times.

(a) $t = 1$ (b) $t = 3$ (c) $t = 5$ (d) $t = 10$

98. ***Fluid Flow*** The emergent velocity v of a liquid flowing from a hole in the bottom of a tank is given by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet per second per second) and h is the depth of the liquid in the tank. Find the rate of change of v with respect to h when (a) $h = 9$ and (b) $h = 4$. (Note that $g = +32$ feet per second per second. The sign of g depends on how a problem is modeled. In this case, letting g be negative would produce an imaginary value for v.)

2.5 **In Exercises 99–104, use implicit differentiation to find *dy/dx*.**

99. $x^2 + 3xy + y^3 = 10$

100. $x^2 + 9y^2 - 4x + 3y = 0$

101. $y\sqrt{x} - x\sqrt{y} = 16$

102. $y^2 = (x-y)(x^2+y)$

103. $x\sin y = y\cos x$

104. $\cos(x+y) = x$

In Exercises 105 and 106, find the equations of the tangent line and the normal line to the graph of the equation at the indicated point. Use a graphing utility to graph the equation, the tangent line, and the normal line.

105. $x^2 + y^2 = 20$, $(2, 4)$

106. $x^2 - y^2 = 16$, $(5, 3)$

2.6

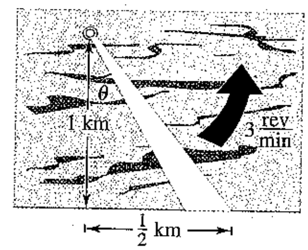
107. A point moves along the curve $y = \sqrt{x}$ in such a way that the y-value is increasing at a rate of 2 units per second. At what rate is x changing for each of the following values?

(a) $x = \frac{1}{2}$ (b) $x = 1$ (c) $x = 4$

108. ***Surface Area*** The edges of a cube are expanding at a rate of 5 centimeters per second. How fast is the surface area changing when each edge is 4.5 centimeters?

109. ***Changing Depth*** The cross section of a 5-meter trough is an isosceles trapezoid with a 2-meter lower base, a 3-meter upper base, and an altitude of 2 meters. Water is running into the trough at a rate of 1 cubic meter per minute. How fast is the water level rising when the water is 1 meter deep?

110. ***Linear and Angular Velocity*** A rotating beacon is located 1 kilometer off a straight shoreline (see figure). If the beacon rotates at a rate of 3 revolutions per minute, how fast (in kilometers per hour) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ kilometer down the shoreline?



Not drawn to scale

111. ***Moving Shadow*** A sandbag is dropped from a balloon at a height of 60 meters when the angle of elevation to the sun is 30° (see figure). Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at a height of 35 meters. [*Hint:* The position of the sandbag is given by $s(t) = 60 - 4.9t^2$.]