Calculus BC—Exam 1

Section I, Part A

Time: 55 minutes
Number of questions: 28

NO CALCULATOR MAY BE USED IN THIS PART OF THE EXAMINATION.

Directions: Solve each of the following problems. After examining the form of the choices, decide which is the best of the choices given.

In this test: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. $\int_0^1 \sqrt{x}(x^2 + 1)\,dx =$

(A) $\frac{4}{3}$ (B) $\frac{9}{7}$ (C) $\frac{16}{15}$

(D) $\frac{20}{21}$ (E) $\frac{4}{21}$

2. If $x = e^{4t}$ and $y = \sin 6t$, then $\frac{dy}{dx} =$

(A) $\frac{3e^{-4t}\cos 6t}{2}$. (B) $-\frac{3\cos 6t}{2e^{4t}}$. (C) $\frac{3e^{-4t}\cos t}{2}$.

(D) $e^{-4t}\cos 6t$. (E) $6\cos 6t$.

3. The function f defined by $f(x) = x^4 - x^2$ has a relative minimum at $x =$

(A) $\sqrt{2}$. (B) 1. (C) $\frac{\sqrt{2}}{2}$.

(D) $\frac{1}{2}$. (E) 0.

4. $\frac{d}{dx}x^2e^{\ln x^3} =$

(A) $6x^3$ (B) $5x^4$ (C) $2x + 3x^2$

(D) $2x^4 + x^5$ (E) $6x^4$

5. If $g(x) = \frac{1}{4}e^{2x-6} + (x-2)^{5/2}$, then $g'(3) =$

(A) 3. (B) $\frac{5}{2}$. (C) $\frac{11}{4}$.

(D) $\frac{1}{4}$. (E) 0.

6. Find the slope of the line normal to the curve $y = -\sqrt{x+4}$ at the point where $x = 0$.

(A) -4 (B) $-\frac{1}{4}$ (C) $-\frac{1}{8}$

(D) $\frac{1}{4}$ (E) 4

7. Compute $\frac{dy}{dx}$ for the relation $2xy^2 + 3\ln y = x^2 - 3y^3$ at the point $(3, 1)$.

(A) $\frac{5}{2}$ (B) $\frac{3}{4}$ (C) $\frac{5}{12}$

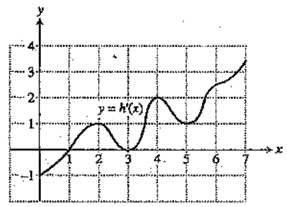
(D) $\frac{1}{8}$ (E) $\frac{1}{6}$

8. $\int_1^{\infty} \frac{x^2}{(1+x^3)^2}\,dx$ is

(A) $-\frac{1}{6}$. (B) $-\frac{1}{24}$. (C) $\frac{1}{24}$.

(D) $\frac{1}{6}$. (E) divergent.

The function $h(x)$ is continuous and differentiable on the domain $[0, 7]$. The graph of $h'(x)$ is shown. Use the graph for Questions 9, 10, and 11.



9. At what value of x does $h(x)$ have its absolute minimum?

(A) 0 (B) 1 (C) 3
(D) 5 (E) 7

10. The point $(5, 2)$ is on the graph of $y = h(x)$. An equation of the line tangent to $h(x)$ at $(5, 2)$ is

(A) $y - 2 = x - 5$. (B) $y = x - 2$.
(C) $y - 2 = 2(x - 5)$. (D) $x = 5$.
(E) $y = 2$.

11. How many inflection points does h have on the interval $(0, 7)$?

(A) 3 (B) 4 (C) 5
(D) 6 (E) 7

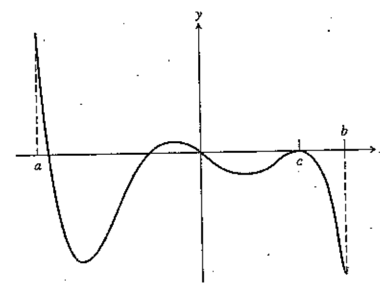
12. The sum of the infinite geometric series $\frac{8}{25} - \frac{24}{125} + \frac{72}{625} - \frac{216}{3125} + \cdots$ is

(A) 0.2. (B) 0.6. (C) 0.8.
(D) 1.0. (E) 1.2.

13. A particle moves along the x-axis so that its acceleration at any time t is $a(t) = 2t - 3$. If the initial velocity of the particle is -4, at what time t in the time interval $0 \le t \le 5$ is the particle farthest left?

(A) 0 (B) $\frac{3}{2}$ (C) 3
(D) 4 (E) 5

14. The graph of $f'(x)$ is shown. It is tangent to the x-axis at point c. Which of the following describes all relative extrema of $f(x)$ on the open interval (a, b)?



(A) One relative maximum and one relative minimum
(B) One relative maximum and two relative minima
(C) Three relative maxima and two relative minima
(D) Two relative maxima and two relative minima
(E) Two relative maxima and one relative minimum

15. The length of the path described by the parametric equations $x = 2 \sin t$ and $y = 3 \cos t$ for $0 \le t \le \pi/2$ is given by

(A) $\int_0^{\pi/2} \sqrt{4 + 5 \sin^2 t}\, dt$.
(B) $\int_0^{\pi/2} \sqrt{4 \cos^2 t - 9 \sin^2 t}\, dt$.
(C) $\int_0^{\pi/2} \sqrt{4 \sin^2 t + 9 \cos^2 t}\, dt$.
(D) $\int_0^{\pi/2} \sqrt{1 + \frac{9 \sin^2 t}{4 \cos^2 t}}\, dt$.
(E) none of the above.

16. $\lim_{x\to 3} \frac{e^{x^2} - e^9}{x - 3} =$

(A) 0 (B) $\frac{e^9}{3}$ (C) $3e^9$

(D) $6e^9$ (E) ∞

17. Let f be the function defined by $f(x) = \ln(x + 4)$. The third-degree Taylor polynomial for f centered about $x = -3$ is

(A) $x - 3 - \frac{(x-3)^2}{2} - \frac{(x-3)^3}{3}$.

(B) $x - 3 - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3}$.

(C) $-3 + x - \frac{(x+3)^2}{2} + \frac{(x+3)^3}{3}$.

(D) $3 + x + \frac{(x+3)^2}{2} + \frac{(x+3)^3}{3}$.

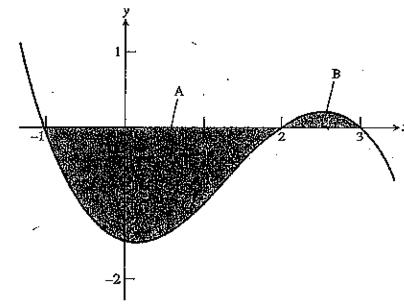
(E) $3 + x - \frac{(x+3)^2}{2} + \frac{(x+3)^3}{3}$.

18. For what values of t does the curve defined by the parametric equations $x = \frac{4}{3}t^3 - t^2$ and $y = t^5 + t^2 - 7t$ have a vertical tangent?

(A) 0 only (B) 0 and $\frac{1}{2}$ (C) $\frac{1}{2}$ only

(D) 1 only (E) 0, $\frac{1}{2}$, and 1

19. The graph of $y = f(x)$ is shown. Let A and B be positive numbers that represent the area of each shaded region. Evaluate $\int_3^{-1} f(x)\,dx + 3\int_{-1}^{2} f(x)\,dx$ in terms of A and B.



(A) $-2A - B$ (B) $2A + B$ (C) $3A - B$

(D) $3A + B$ (E) $-3A - B$

20. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n \cdot 4^n}$ converges?

(A) $-\frac{3}{2} \le x \le \frac{5}{2}$ (B) $-\frac{3}{2} \le x < \frac{5}{2}$

(C) $-\frac{3}{2} < x \le \frac{5}{2}$ (D) $-3 \le x < 5$

(E) $-3 < x \le 5$

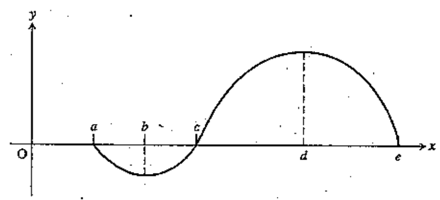
21. The expression representing the area inside one leaf of the polar rose $r = 3\cos 2\theta$ is given by

(A) $\int_0^{\pi/4}\sqrt{1 + 9\sin^2 2\theta}\,d\theta$ (B) $\int_0^{\pi/2}\sqrt{1 + 9\sin^2 2\theta}\,d\theta$

(C) $\int_0^{\pi/2} 9\cos^2 2\theta\,d\theta$ (D) $\frac{1}{2}\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta$

(E) $\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta$

22.

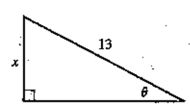


The graph of f is shown. If $h(x) = \int_a^x f(t)\,dt$, for what value of x does $h(x)$ have its minimum?

(A) a (B) b (C) c

(D) d (E) e

23. In the right triangle shown, θ is increasing at a constant rate of 2 radians per minute. In units per minute, at what rate is x increasing when $x = 12$?



(A) 2 (B) 4 (C) 5

(D) 10 (E) 24

24. The Taylor series for $\cos x$ about $x = 0$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$. If h is a function such that $h'(x) = \cos x^3$, then the coefficient of x^7 in the Taylor series for $h(x)$ about $x = 0$ is

(A) $-\frac{1}{14}$. (B) $-\frac{1}{7!}$. (C) 0.

(D) $\frac{1}{7!}$. (E) $\frac{1}{14}$.

25. The closed interval $[w, v]$ is partitioned into k equal subintervals, each with width Δx, by the numbers $x_0, x_1, \ldots, x_k$ with $w = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = v$. The $\lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{\sqrt{x_j}} \Delta x$ equals

(A) $\sqrt{v} - \sqrt{w}$. (B) $2\left(\sqrt{v} - \sqrt{w}\right)$.

(C) $\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{w}}$. (D) $2\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{w}}\right)$.

(E) $\frac{2}{3}\left(v^{3/2} - w^{3/2}\right)$.

26. $\int \frac{6x - 8}{(x - 3)(x + 2)}\,dx =$

(A) $2\ln|x - 3| + 4\ln|x + 2| + C$

(B) $2\ln|x - 3| + 2\ln|x + 2| + C$

(C) $2\ln|x + 3| + 4\ln|x - 2| + C$

(D) $6x\ln|x - 3| + 8\ln|x + 2| + C$

(E) $4\ln|x - 3| + 2\ln|x + 2| + C$

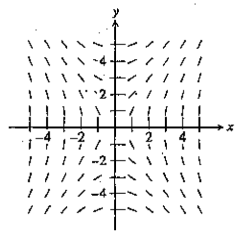
27. $\int 2x\cos x\,dx =$

(A) $x^2\sin x + C$ (B) $x^2\cos\frac{x^2}{2} + C$

(C) $2\sin x - 2x\cos x + C$ (D) $-2x\sin x - 2\cos x + C$

(E) $2x\sin x + 2\cos x + C$

28. Which of the following equations has the slope field shown?



(A) $\frac{dy}{dx} = 2x$ (B) $\frac{dy}{dx} = 2y$ (C) $\frac{dy}{dx} = \frac{2x}{y}$

(D) $\frac{dy}{dx} = xy$ (E) $\frac{dy}{dx} = \frac{2y}{x}$

Calculus BC—Exam 1
Section I, Part B

Time: 50 Minutes
Number of Questions: 17

A GRAPHING CALCULATOR IS REQUIRED FOR SOME QUESTIONS IN THIS PART OF THE EXAMINATION.

Directions: Solve each of the following problems. After examining the form of the choices, decide which is the best of the choices given.

In this test:

1. The exact numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.
2. Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

29. Which of the following sequences converge?

I. $\left\{\frac{2^n}{n+1}\right\}$

II. $\left\{\frac{3n+7}{8n}\right\}$

III. $\left\{\frac{e^{2n}}{e^n+3^n}\right\}$

(A) I only (B) II only (C) I and II only

(D) II and III only (E) None of them

30. When the region enclosed by the graphs of $y = 2x$ and $y = 6x - x^2$ is revolved around the y-axis, the volume of the solid generated is given by

(A) $\pi\int_0^4 \left(8x^2 - 2x^3\right) dx$.
(B) $2\pi\int_0^{16} x\left(4x - x^2\right) dx$.
(C) $2\pi\int_0^4 \left(4x - x^2\right) dx$.
(D) $\pi\int_0^4 \left[\left(6x - x^2\right)^2 - \left(2x\right)^2\right] dx$.
(E) $2\pi\int_0^4 x\left(x^2 - 4x\right) dx$.

31. $\lim_{h \to 0} \dfrac{\ln\left(\frac{1}{e} + h\right) + 1}{h}$ is

(A) $f'(e)$ where $f(x) = \frac{1}{x}$.
(B) $f'(e)$ where $f(x) = -\ln\frac{1}{x}$.
(C) $f'(1)$ where $f(x) = \ln\frac{x}{e}$.
(D) $f'\left(\frac{1}{e}\right)$ where $f(x) = \ln(x + e)$.
(E) $f'\left(\frac{1}{e}\right)$ where $f(x) = \ln x$.

32. The position of an object oscillating on the x-axis is given by $x(t) = 2\sin 4t - \cos 4t$, where t is the time in seconds. In the first 5 seconds, how many times is the velocity of the object equal to 0?

(A) 0
(B) 4
(C) 5
(D) 6
(E) 7

33. Let h be the function defined by $h(x) = \cos 3x + \ln 4x$. What is the least value of x at which the graph of h changes concavity?

(A) 1.555
(B) 0.621
(C) 0.371
(D) 0.096
(E) 0.004

34. Let f be a continuous function on the closed interval $[-2, 5]$. If $f(-2) = 3$ and $f(5) = -7$, then the Intermediate Value Theorem guarantees that

(A) $-7 \le f(x) \le 3$ for all x between -2 and 5.
(B) $f'(c) = -\frac{10}{7}$ for at least one c between -2 and 5.
(C) $f(c) = -3$ for at least one c between -2 and 5.
(D) $f(c) = 0$ for at least one c between -7 and 3.
(E) $f(x)$ is continually decreasing between -2 and 5.

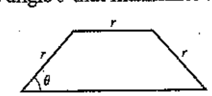
35. Which of the following is the closest to the lowest value of x in the interval $0 \le x \le 6$ such that $\int_0^x (t^2 - 3t)\,dt \le \int_2^x t\,dt$?

(A) 0
(B) 1.1075
(C) 1.663
(D) 1.745
(E) 5.823

36. If $\frac{dy}{dx} = (\ln x + 2)y$ and $y = 2$ when $x = 1$, then $y =$

(A) $2e^{x\ln x}$.
(B) $e^{x + x\ln x}$.
(C) $e^{2 + x\ln x}$.
(D) $2e^{(1/x + 2x - 3)}$.
(E) $\sqrt{4x + 4 + 2x\ln x}$.

37. The isosceles trapezoid shown has legs and the top base of length r. The acute angle θ that maximizes the area of the trapezoid is



(A) 15°.
(B) 30°.
(C) 45°.
(D) 60°.
(E) 75°.

38. Let f be a twice-differentiable function such that $f(2) = 8$ and $f(4) = 5$. Which of the following must be true for the function f on the interval $2 \le x \le 4$?

I. The average value of f is $\frac{13}{2}$.

II. The average rate of change of f is $-\frac{3}{2}$.

III. The average value of f' is $-\frac{3}{2}$.

(A) II only (B) III only (C) I and II only
(D) II and III only (E) I, II, and III

39. Find all values c that satisfy the Mean Value Theorem for the function $f(x) = \frac{2}{(1 + x^2)}$ on the interval $[-1, 2]$.

(A) 0.050 (B) −0.050
(C) 0.102 and 1.801 (D) 0.050 and 2.449
(E) None exist in the interval.

40. The base of a solid is the region in the first quadrant enclosed by the graph of $y = 9 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a base of a rectangle and each height is three times the base, the volume of the solid is given by

(A) $\int_0^9 (9 - y)\, dy$. (B) $3\int_0^9 (9 - x^2)^2\, dx$.
(C) $3\int_0^9 (9 - y)\, dy$. (D) $3\int_0^3 (9 - x^2)^2\, dx$.
(E) $3\int_0^9 (9 + y)\, dy$.

41. Let $f(x) = \int_0^{2x^2} \cos t\, dt$. At how many points in the closed interval $[0, \sqrt{\pi}]$ does the instantaneous rate of change of f equal the average rate of change of f on that interval?

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

42. If h is an antiderivative of $g(x) = \frac{x^3}{1 + x^5}$ and $h(1) = 2$, then $h(3) =$

(A) 4.407. (B) 2.555. (C) 1.852.
(D) 0.555. (E) −0.703.

43. A force of 12 pounds stretches a spring 3 inches beyond its natural length. Assuming Hooke's Law applies, how much work is done stretching the spring from its natural length to 6 inches beyond its natural length?

(A) 4 inch-pounds (B) 8 inch-pounds
(C) 36 inch-pounds (D) 54 inch-pounds
(E) 72 inch-pounds

44. The length of the polar curve $r = 1 - 2\cos 2\theta$ is

(A) 6.925. (B) 10.008. (C) 13.365.
(D) 17.629. (E) 20.016.

45. If $\frac{dy}{dx} = x + y$ and $f(0) = 2$, use Euler's method with $\Delta x = 0.5$ to approximate $f(1)$.

(A) 1.75 (B) 3 (C) 4
(D) 4.75 (E) 5.75

FREE RESPONSE

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS IN THIS PART OF THE EXAMINATION.

1. A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^2 + 4} \quad \text{and} \quad \frac{dy}{dt} = 3e^t + 2e^{-t}$$

for all real values of t. At time $t = 0$, the position of the particle is $(3, 4)$.

(A) Find the speed and acceleration vector of the particle at time $t = 0$.

(B) Find the equation of the line tangent to the path of the particle at time $t = 0$.

(C) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$.

(D) Find the x-coordinate of the position of the particle at time $t = 2$.

2. Let $P(x) = 8 - 4(x-3) + 5(x-3)^2 - 7(x-3)^3 + 9(x-3)^4 - 6(x-3)^5$ be the fifth-degree Taylor polynomial for the function f about 3. Assume f has derivatives of all orders for all real numbers.

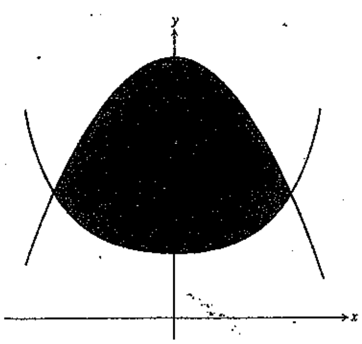
(A) Find $f(3)$ and the value of the fourth derivative, $f^{IV}(3)$.

(B) Write the third-degree Taylor polynomial for f' about 3, and use it to approximate $f'(3.2)$.

(C) Write the sixth-degree Taylor polynomial for $h(x) = \int_3^x f(t)\,dt$ about 3.

(D) Can $f(4)$ be determined from the given information? Explain.

3. Let R be the region enclosed by the graphs of $y = 4 - \frac{1}{2}x^2$ and $y = \sec\frac{x}{2}$.



(A) Find the area of R.

(B) Find the volume of the solid generated when R is rotated about the x-axis.

(C) Write an expression involving one or more integrals that gives the length of the boundary of region R. DO NOT EVALUATE.

NO CALCULATOR MAY BE USED IN THIS PART OF THE EXAMINATION.

4. Consider the curve given by $4x - 2x^2y = y^2 + 1$.

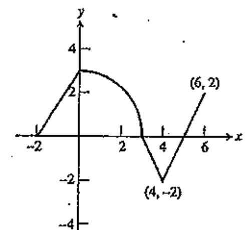
(A) Show that $\frac{dy}{dx} = \frac{2 - 2xy}{y + x^2}$.

(B) Show there is a point Q with x-coordinate 1 such that there is a horizontal tangent to the curve at Q.

(C) Evaluate $\frac{d^2y}{dx^2}$ at point Q. Is there a local minimum, local maximum, or neither at Q? Justify your answer.

5. The graph of g shown in the figure consists of a quarter-circle and three line segments. Let h be the function defined by

$$h(x) = \int_0^x g(t)\,dt.$$



(A) Evaluate $h(4)$.

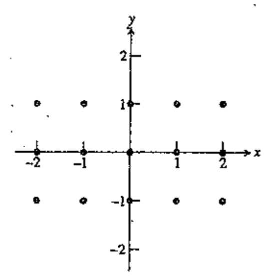
(B) Find all values of x in the interval $[-2, 6]$ at which h has a relative minimum. Justify your answer.

(C) Find the value of $h'(2)$.

(D) Find the x-coordinate of each point of inflection of the graph of h on the interval $(-2, 6)$. Justify your answer.

6. Consider the differential equation $\frac{dy}{dx} = x^2(1 - y)^2$.

(A) On the axes provided, sketch a slope field for the given differential equation at the 15 points indicated.



(B) Find the particular solution $y = f(x)$ to the given differential equation if $f(3) = 0$.

(C) For $f(x)$ found in part (B), find $\lim_{x\to\infty} f(x)$. Explain how the answer is related to a characteristic of the slope field.